КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 675—678

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p.675-678

ON THE ESTIMATION OF ORTHOGONAL COEFFICIENTS

G.M. Phillips and P.J. Taylor

1. Introduction. S.N. Bernstein [1] gave the following estimate of the minimax error, proving that, for each $f \in C^{n+1}[-1,1]$, $\exists\ \xi \in (-1,1)$ such that

$$E_n(f) = \inf_{q \in P_n} \|f-q\|_\infty = \frac{|f^{(n+1)}(\xi)|}{2^n(n+1)!}, \tag{1}$$

where $\|\cdot\|_\infty$ denotes the maximum norm and P_n denotes the subspace of polynomials of degree at most n. This result can also be expressed as

$$E_n(f) = \frac{|f^{(n+1)}(\xi)|}{(n+1)!} E_n(x^{n+1}), \quad -1 < \xi < 1. \tag{2}$$

Phillips [5] showed that (2) holds, more generally, for the error of the best approximant with respect to the p-norm,

$$E_n(f) = \inf_{q \in P_n} \|f-q\|_p, \quad 1 \leq p \leq \infty.$$

More recently, some interest has been shown in finding projections $P : C^{n+1}[-1,1] \to P_n$ so that

$$\|f-Pf\|_\infty = \frac{|f^{(n+1)}(\xi)|}{2^n(n+1)!}, \quad -1 < \xi < 1. \tag{3}$$

In view of (1) such projections can be thought of as near-minimax. The most obvious of such projections is where $(Pf)(x)$ is the interpolating polynomial for f on the zeros of the Chebyshev polynomial T_{n+1}. The verification of this case follows easily from the Cauchy form of the error of the interpolating polynomial.

Note that, for any projection $P : C^{n+1}[-1,1] \to P_n$, it follows immediately from (1) that

$$\|f - Pf\|_\infty \geq E_n(f) \geq \frac{1}{2^n(n+1)!} \cdot \min_{-1\leq x\leq 1} |f^{(n+1)}(x)|.$$

Thus, in order to verify (3), it suffices to show that

$$\|f - Pf\|_\infty \leq \frac{1}{2^n(n+1)!} \|f^{(n+1)}\|_\infty.$$

A second projection satisfying (3) was found by Phillips and Taylor [6]; this is the projection Pf such that f - Pf equioscillates on the point set consisting of the n + 2 extreme points of T_{n+1}. This projection is commonly recommended as a first iterate in the Remez algorithm for computing the minimax polynomial.

Very recently Elliott, Paget, Phillips and Taylor [4] have discovered other projections which satisfy (3). Of these, the most important is the truncated Chebyshev series projection; that is, we define

$$(Pf)(x) = \sum_{r=0}^{n}{}' a_r T_r(x),$$

where $\sum'$ denotes a summation whose first term is halved and

$$a_r = \frac{2}{\pi}\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} f(x)\, T_r(x)\, dx.$$

In the next section we derive an estimate of the coefficients of a general orthogonal polynomial series. In the special case of the Chebyshev series, we find that the modulus of the coefficient a_{n+1} also has the form of the right side of (3).

2. Orthogonal coefficients. Let $\omega(x)$ denote a non-negative weight function on [-1,1] such that

$$\int_{-1}^{1} \omega(x)\, x^n\, dx$$

exists for n = 0,1,... . We write $p_0, p_1, \ldots$ to denote the system of polynomials which are orthogonal with respect to ω on [-1,1], normalized so that each polynomial has leading coefficient 1; that is, $p_n(x) - x^n \in P_{n-1}$. Thus the orthogonal coefficient is given by

$$c_{n+1} = (f, p_{n+1})/(p_{n+1}, p_{n+1}), \tag{4}$$

where

$$(f,g) = \int_{-1}^{1} \omega(x)\, f(x)\, g(x)\, dx.$$

By orthogonality, we have

$$(f,p_{n+1}) = (f-q,p_{n+1}) \tag{5}$$

for any $q \in P_n$. In particular, let us choose q as the polynomial $\in P_n$ which interpolates f on the zeros of p_{n+1}. Then, from the Cauchy error formula of interpolation, we have

$$f(x) - q(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} p_{n+1}(x). \tag{6}$$

It follows immediately from (4), (5), (6) and the mean value theorem of integration that

$$c_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}, \quad -1 < \xi < 1. \tag{7}$$

To facilitate comparison of the size of $\| c_{n+1} p_{n+1} \|_\infty$ for different orthogonal series, it is now convenient to re-normalize the orthogonal polynomials so that each polynomial has maximum norm unity. To underline this change we will write $a_0, a_1, \ldots$ to denote the orthogonal coefficients of the re-normalized series. It follows from (7) that the re-scaled coefficients satisfy

$$a_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!} \| p_{n+1} \|_\infty, \quad -1 < \xi < 1, \tag{8}$$

where, as before, the orthogonal polynomial p_{n+1} has leading coefficient 1.

The smallest value of $\| p_{n+1} \|_\infty$ is $1/2^n$, attained by the Chebyshev series, where $\omega(x) = (1-x^2)^{-\frac{1}{2}}$. For this special case, Elliott [2] deduced (8) from (4), on replacing the Chebyshev polynomial T_{n+1} by Rodrigue's formula and integrating by parts n+1 times. Elliott [3] has also used complex variable methods to estimate Chebyshev coefficients.

Finally, in view of the fact (see (8) and (1)) that the modulus of the Chebyshev coefficient a_{n+1} and $E_n(f)$ both satisfy expressions of the form

$$\frac{|f^{(n+1)}(\xi)|}{2^n (n+1)!},$$

for possibly different values of ξ, it is interesting to note a result of Rivlin [7],

$$|a_{n+1}| \leqslant \frac{4}{\pi} E_n(f). \tag{9}$$

Following Rivlin's method [7], we may readily extend (9) from the special case of Chebyshev series to a general orthogonal polynomial series.

We obtain

$$|a_{n+1}| \leq K_n \cdot E_n(f),$$

where

$$K_n = \|p_{n+1}\|_\infty \int_{-1}^{1} \omega(x)\,|p_{n+1}(x)|\,dx \Big/ (p_{n+1}, p_{n+1}).$$

References

1. S.N. Bernstein. Leçons sur les propriétés extrémales et al meilleure approximation des fonctions analytiques d'une variable réelle. Gauthier-Villars, Paris, 1926.
2. D. Elliott. A Chebyshev series method for the numerical solution of Fredholm integral equations. Computer J., Vol.6, 1963, p.102.
3. D. Elliott. The evaluation and estimation of the coefficients in the Chebyshev series expansion of a function. Math. Comp., Vol.18, 1964, p.274.
4. D. Elliott, D.F. Paget, G.M. Phillips and P.J. Taylor. Error of truncated Chebyshev series and other near minimax polynomial approximations. J. Approx. Theory (submitted).
5. G.M. Phillips. Error estimates for best polynomial approximations. Approximation Theory, ed. A. Talbot. Academic Press, London, 1970, p.1.
6. G.M. Phillips and P.J. Taylor. Polynomial approximation using equioscillation on the extreme points of Chebyshev polynomials. J. Approx. Theory, Vol.36, 1982, p.257.
7. T.J. Rivlin. The Chebyshev Polynomials. Wiley, New York, 1974.

The Mathematical Institute,
University of St Andrews,
St Andrews, Scotland.

Department of Mathematics,
University of Stirling,
Stirling, Scotland.