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AGAIN ON MARKOV'S INEQUALITY

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1. Introduction. By its importance for the polynomial approximation theory, the classical Markov inequality is still the object of numerous investigations. As far as the theory is well developed in one dimensional case (see e.g. [7]) incomparably less has been done in the case of several variables. Versions of Markov's inequality obtained by means of Siciak's extremal function methods, that contain in particular the case of convex sets (see [6] and [9]), seem to be most satisfactory in this matter. However, the case of sets with edges still affords difficulties and a first attempt in this direction was recently made by Goetgheluck [2] who gave estimates of derivatives of a polynomial of two variables on the set $E^p = \{(x,y) \in \mathbb{R}^2;\ 0 \leq x \leq 1,\ 0 \leq y \leq x^p\}$ where $p \geq 1$. Having been inspired by Goetgheluck's example we propose in this note a method based on the known extremal function technics which permits us to give Markov's type estimates in more general cases including that of subanalytic sets in $\mathbb{R}^N$.

2. Main result. If E is a subset of the space $\mathbb{R}^N$, let c be a point of the closure $\overline{E}$ of E. Assume that there is a polynomial map $w = (w_1,\dots,w_N) : \mathbb{R} \to \mathbb{R}^N$ of degree d such that $w((0,1]) \subset E$, $w(0) = c$ and for each $t \in (0,1]$ one can find $\varrho > 0$ and a cube I of edge 2ϱ such that $w(t) \in I \subset E$. Denote by $\varrho(t)$ the supremum spread over all such ϱ and set

$$r(t) = \inf\{\varrho(s) : t \leq s \leq 1\}$$

for $t \in (0,1]$. Under these assumptions we can prove

Proposition 2.1. For each polynomial p in $\mathbb{R}^N$ of degree at most n and each multiindex $\alpha = (\alpha_1, \dots, \alpha_N)$, we have

$$|D^\alpha p(c)| \leqslant 2e^{2d}[n^2/r(1/n^2)]^{|\alpha|} \|p\|_E ,$$

where $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}$, $|\alpha| = \alpha_1 + \dots + \alpha_N$ and $\|p\|_E =$ $\sup |p|(E)$.

P r o o f. We shall make use of the extremal function Φ_K associated with a compact set K in the space $\mathbb{C}^N$ (see [8]),

$$\Phi_K(z) = \sup\{|p(z)|^{1/\deg p} ; p \text{ is a polynomial of degree} \geqslant 1 \text{ with } \|p\|_K \leqslant 1\}$$

for $z \in \mathbb{C}^N$. It is known that

(1) $$\Phi_{[-1,1]}(z) = |z + \sqrt{z^2 - 1}|, \quad z \in \mathbb{C},$$

where the branch of the root is so chosen that for $x > 1$ we have $\Phi_K(x) > 1$. Hence one can easily check that for each $t \in [0,1)$

(2) $$\Phi_{[t,1]}(0) = (1 + \sqrt{t})/(1 - \sqrt{t}) .$$

For each $s \in (0,1]$, choose a cube I_s of edge $2r(s)$ such that $w(s) \in I_s \subset E$ and for fixed $t \in (0,1]$, set

$$E_t = \overline{\bigcup_{s \in [t,1]} I_s} .$$

By the definition of the extremal function, from (2) we derive

$$\Phi_{E_t}(c) \leqslant [\Phi_{[t,1]}(0)]^d = [(1 + \sqrt{t})/(1 - \sqrt{t})]^d .$$

If now p is a polynomial in $\mathbb{R}^N$ of degree $\leqslant n$ then for each $\alpha \in \mathbb{N}_0^N$, we have

$$|D^\alpha p(c)| \leqslant \|D^\alpha p\|_{E_t} [\Phi_{E_t}(c)]^n ,$$

whence by putting $t_n = 1/n^2$ we get

(3) $$|D^\alpha p(c)| \leqslant 2e^{2d} \|D^\alpha p\|_{E_{t_n}} = 2e^{2d} |D^\alpha p(b)|$$

with a suitably chosen point $b \in E_{t_n}$. Take now a sequence $\{b^\nu\}$ in the set $\bigcup_{s \in [t_n,1]} I_s$ such that $b^\nu \to b$ as $\nu \to \infty$, and a sequence $\{s_\nu\} \subset [t_n, 1]$ such that $b^\nu \in I_{s_\nu}$ for each ν. By the classical Markov inequality for the interval $[-1,1]$, for each ν, we get

$$|D^\alpha p(b^\nu)| \leq [n^2/r(s_\nu)]^{|\alpha|} \, \| p \|_{I_{s_\nu}} \leq [n^2/r(t_n)]^{|\alpha|} \, \| p \|_E ,$$

whence by (3) we obtain the assertion of the proposition.

A better estimate of derivatives of a polynomial can be obtained for subsets of the space $\mathbb{C}^N$ with nonvoid interior. By a recent result of Tung [11], if $E = \{z \in \mathbb{C}^N ; |z| \leq 1\}$, then for each polynomial $p : \mathbb{C}^N \to \mathbb{C}$ of degree $\leq n$ and each $\alpha \in \mathbb{N}_0^N$,

$$\| D^\alpha p \|_E \leq n^{|\alpha|} \, \| p \|_E .$$

Starting with this inequality we can prove the following

<u>Proposition 2.2.</u> Suppose E is a subset of $\mathbb{C}^N$ and $c \in \overline{E}$. Assume there is a polynomial map $w : \mathbb{C} \to \mathbb{C}^N$ of degree d such that $w((0,1]) \subset E$ and $w(0) = c$ and for each $t \in (0,1]$ one can find a point $a(t) \in E$ such that $w(t) \in B(a(t), r(t)) \subset E$, with an increasing positive function r defined on $(0,1]$, $B(a,r)$ denoting the ball $\{z \in \mathbb{C}^N ; |z - a| \leq r\}$. Then for each polynomial p in $\mathbb{C}^N$ of degree $\leq n$ and each $\alpha \in \mathbb{N}_0^N$,

$$|D^\alpha p(c)| \leq 2e^{2d} [n/r(1/n^2)]^{|\alpha|} \, \| p \|_E .$$

<u>3. Case of subanalytic sets.</u> The case of special interest is that of the function r of Proposition 2.1 admitting a polynomial bound from below on $(0,1]$. It appears that E being subanalytic meets this requirement. We recall that a subset E of $\mathbb{R}^N$ is said to be <u>semianalytic</u> if for each point $c \in \mathbb{R}^N$ there exists a neighborhood U of c and a finite number of real analytic functions f_{ij} and g_{ij} defined in U such that

$$E \cap U = \bigcup_i \bigcap_j \{f_{ij} > 0, \, g_{ij} = 0\}$$

(see [4]). E is said to be <u>subanalytic</u> if for each point $c \in \mathbb{R}^N$, there is a neighborhood U of c such that $E \cap U = \pi(A)$ where A is a bounded semianalytic set in $\mathbb{R}^{N+M}$ and $\pi : \mathbb{R}^N \times \mathbb{R}^M \ni (x,y) \to x \in \mathbb{R}^N$ is the natural projection (see [1]). (For properties of semianalytic and subanalytic sets we refer to [4], [1] and [3].) For open subanalytic sets, the following lemma improves a known Bruhat-Cartan-Wallace theorem (see [1], "curve selecting lemma") as well as permits us to give a refinement of Proposition 2.1. The proof of the lemma is owed to W. Pawłucki (personal communication).

Lemma 3.1. Let E be an open subanalytic set in $\mathbb{R}^N$ and let $c \in \overline{E}$. Then there is a polynomial map $w : \mathbb{R} \to \mathbb{R}^N$ such that $w((0,1]) \subset E$ and $w(0) = c$.

P r o o f. By the curve selecting lemma (see [1]) there is a semianalytic arc $\lambda : [0,1] \to E \cup \{c\}$ such that $\lambda(0) = c$. On the other hand, by Puiseux's theorem (see e.g. [5]) there is an integer $m > 0$ such that the function $\psi : t \to \lambda(t^m)$ admits an analytic extension to a neighborhood $[-\delta, \delta]$ of $0 \in \mathbb{R}$. By the regular separation theorem for subanalytic sets (see [3], p.9.5 or else [10] theorem 3.1), there exists a positive constant C and a positive integer n such that

(4) $$\operatorname{dist}(\psi(t), \mathbb{R}^N \setminus E) \geqslant C|\psi(t) - c|^n$$

for $t \in [0, \delta]$, $|\cdot|$ denoting the Euclidean norm in $\mathbb{R}^N$. By the analyticity of ψ, we have

(5) $$|\psi(t) - c| \geqslant C' t^m$$

for $t \in [0, \delta]$, with some constant $C' > 0$ and some integer $m > 0$. Set $\varphi = T_0^p \psi$, the p-th Taylor's sum of ψ at 0, where $p = mn$. Then

$$|\varphi(t) - \psi(t)| \leqslant M t^{p+1}$$

for $t \in [0, \delta]$ with an appropriate constant $M > 0$, whence by (4) and (5),

(6) $$\operatorname{dist}(\varphi(t), \mathbb{R}^N \setminus E) > M' t^p$$

for $t \in (0, \mu]$ with $\mu > 0$ sufficiently close to 0 and some $M' > 0$. Thus, $\varphi(t) \in E$ for $t \in (0, \mu]$ and $\varphi(0) = c$. Therefore the map $w(s) = \varphi(\mu s)$, $s \in \mathbb{R}$, fulfils the assertion of the lemma.

Since for each subanalytic set $E \subset \mathbb{R}^N$ the set $\operatorname{int} E$ is also subanalytic, by (6) and Proposition 2.1, we get

Corollary 3.2. Let E be a subanalytic subset of $\mathbb{R}^N$. Then for each point $c \in \operatorname{int} E$ there is a constant $d > 0$ such that for each polynomial p in $\mathbb{R}^N$ of degree $\leqslant n$ and each $\alpha \in \mathbb{N}_0^N$,

$$|D^\alpha p(c)| \leqslant e^d n^{d|\alpha|} \|p\|_E .$$

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