КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 684—689

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p.684-689

SOME NEW CLASSES OF PERFECT BANACH LATTICES RELATED TO THE CARLEMAN OPERATORS AND THEIR ADJOINTS

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In this paper an abstract version of Carleman operators (see also [2]) is considered. We characterize also the operators whose Köthe adjoints are Carleman operators. These notions enable us to introduce two kinds of perfect Banach lattices called further "almost (AM)-spaces" and "almost (AL)-spaces" respectively. Some characterizations of these two classes of Banach lattices are given, and moreover it is showed that they are dual each other. It is also to be remarked that the method to define the second class of operators can be applied to every ideal operator acting between Banach lattices in order to get new interesting operator classes.

For the unexplained terminology we follow [4].

Let E be an Archimedean vector lattice. Then there exists an order complete vector lattice E^u such that each family of pairwise disjoint elements of E^u is order bounded and such that E can be identified with an order dense vector sublattice of E^u. For more details concerning E^u see [1], [6], [8]. An immediate consequence of the construction of E^u is the fact that for each $f \in E^u$ there exists a maximal orthogonal system $(e_i)_{i \in I}$ in E such that $P_{e_i} f \in E$ for all $i \in I$. E^u is the space $L_o(X, \mu)$ of all equivalence classes of μ-a.e. finite μ-measurable functions whenever E is a Köthe function space on (X, μ).

Let E be a Banach lattice and let $f \in E^*$. f is called an (o)-<u>continuous functional</u> if from a net $x_i \downarrow 0$, $x_i \in E$, it follows $f(x_i) \to 0$ The set of all (o)-continuous functionals on E is denoted by E^x, which can be reduced to zero. If $E^x \neq \{0\}$, then the evaluation map $k: E \longrightarrow E^{xx}$ is a lattice homomorphism. E is called a <u>perfect Banach lattice</u> if k is a surjective isometry. Each maximal Köthe function space on a finite measure space is a perfect Banach lattice [4].

Definition 1 [2] Let E be a Banach space and let F be an Archimedean vector lattice. Then $T:E \longrightarrow F$ is called a Carleman operator if T maps the unit ball of E into an order bounded subset of F^u.

This definition coincides with the classical one (see [5]) whenever E and F are Köthe function spaces and moreover E has an order continuous norm.

The equivalence of 1), 3), 5) in the following theorem is known. (See [2]).

Theorem 2 Let E be a Banach space and F be a perfect Banach lattice. Then the following assertions are equivalent:

1) $T:E \longrightarrow F$ is a Carleman operator.

2) Let $1 \leq p < \infty$ be a fixed number. Then there exists $f \in (F^u)_+$ such that for every $x_i \in E$, $i = 1,2,\ldots,n$, we get

$$\Big(\sum_{i=1}^{n} |Tx_i|^p\Big)^{1/p} \leq \Big[\sup_{\|x^*\| \leq 1} \Big(\sum_{i=1}^{n} |x^*(x_i)|^p\Big)^{1/p}\Big] f.$$

3) There exists a maximal orthogonal system $(e_i)_{i\in I}$ in E such that $P_{e_i}T$ are strongly majorizing operators for all $i \in I$. (An operator $U:E \longrightarrow F$ is strongly majorizing operator if there exists $f \in F_+$ such that $|Ux| \leq \|x\|_E f$.)

4) There exist $e_i \in E$ as above such that all $P_{e_i}T$ are majorizing operators. ($U:E \longrightarrow F$ is a majorizing operator if there exists $f \in F^{**}$ such that $|Ux| \leq \|x\|_E \cdot f$.)

5) For every norm converging to zero sequence $x_n \in E$ it follows that $Tx_n \xrightarrow{\sigma} 0$ in F^u. ($y_n \xrightarrow{\sigma} 0$ if there exist $z_n \in F^u$ such that $z_n \downarrow 0$ and $|y_n| \leq z_n$, $n \in N$.)

Proof 1)$\Longleftrightarrow$2) is easy. 1)$\Longrightarrow$3) follows by the remark made after the introduction of F^u and 3)$\Longrightarrow$1) is an easy consequence of the definition of F^u. 3)$\Longrightarrow$4) is obvious. 4)$\Longrightarrow$1) Let's recall that, for a majorizing operator $U:E \longrightarrow F$ the expression $m_U :=$ $:= \inf\{m > 0;\ \|\sup_{j\leq n} Ue_j\| \leq m \cdot \sup_{j\leq n} \|e_j\|,\ n \in N\}$ is finite. Denote by $y_{i,H}$ the element $\sup_{e\in H} |P_{e_i}Te| \in F$, where H is a finite subset of the unit ball B of E and $i \in I$ is fixed. Then $(y_{i,H})_H$ is an upward directed family of elements of $m_i V^{xx}$, where V^{xx} is the unit ball of F^{xx} and $m_i = m_{P_{e_i}T} < \infty$. Since F^{xx} is a perfect Banach lattice, by Theorem 2.1-- [7], it follows that there exists $y_i = \sup_H y_{i,H} \in F^{xx}$. It is clear that each maximal orthogonal system $(e_i)_{i\in I}$ of F is also maximal in F^u. Consequently, denoting by $\widetilde{P}_{e_i}$ the band projection on the band generated in F^u by e_i, we get $kP_{e_i}T = \widetilde{P}_{e_i}kT$. But $|kP_{e_i}Te| \leq \|e\| y_i$ for all $i \in I$, hence $\widetilde{P}_{e_i}kT : E \longrightarrow F^{xx}$ is a strongly majorizing operator for all $i \in I$, and by $3) \Longrightarrow 1)$ for kT it follows that kT is a Carleman operator. Since $F^u = (F^{xx})^u$, it follows that T itself is a Carleman operator, $1) \Longrightarrow 5)$ is obvious. $5) \Longrightarrow 1)$ By definition of Carleman operators it's clear that it suffices to show that $P_{e_i}T$ are Carleman operators for every maximal orthogonal system $(e_i)_{i \in I}$ in F. Then by thm. 2.3 - [7] we are reduced to the case F is a Köthe function space on a finite measure space (X, μ). Then the implication $5) \Longrightarrow 1)$ was proved in [5]. ∎

<u>Definition 3</u> Let E be a perfect Banach lattice. We say that E is an <u>almost (AM)-space</u> (abbreviated a(AM)-space) if Id_E is a Carleman operator.

Perfect (AM)-spaces are a(AM)-spaces. Moreover
$E := \{ f \in L_o(0,1) ; \ |||f||| = (\sum_{n=2}^{\infty} 1/n^2 \|P_{A_n} f\|_{\infty}^2)^{1/2} < \infty$, where $A_n = (1/n-1, 1/n) \}$ is a maximal Köthe function space which is an a(AM)-space non-isomorphic to an (AM)-space. Also each purely atomic perfect Banach lattice E is an a(AM)-space. (See [3]). On the other side $L_p(0,1)$ with $1 \leq p < \infty$ are not a(AM)-spaces.

From two a(AM)-spaces we get a new a(AM)-space using the notion of perfect M-tensor product of two perfect Banach lattices, notion due to D. Vuza [7].

Let E_1, E_2 be two perfect Banach lattices and let $\mathcal{L}_m^x(E_1^x, E_2)$ the space of all (o)-continuous operators from E_1^x into E_2 such that $B := U\{x \in E_1^x ; \|x\| \leq 1\}$ is an order bounded subset of E_2, endowed with $\|U\|_m = \|\sup B\|$. Then $\mathcal{L}_m^x(E_1^x, E_2)$ becomes a perfect Banach lattice which contains the subspace $\mathcal{F}^x(E_1^x, E_2)$ of all (o)-continuous finite rank operators. The closure of this latter space in the topology of uniform convergence on order bounded subsets of $[\mathcal{L}_m^x(E_1^x, E_2)]^x$ is called the <u>perfect M-tensor product of E_1 and E_2</u> and it is denoted by $E_1 \hat{\otimes}_M E_2$.

<u>Proposition 4</u> 1) Let F be a projection band of an a(AM)-space E. Then F is an a(AM)-space too.

2) If E_i, $i = 1,2$ are a(AM)-spaces, then $E_1 \hat{\otimes}_M E_2$ is an a(AM)-space.

We omit the standard proof.

In what follows we assume that F is a maximal Köthe function space. Let's denote by $C(E,F)$ the space of all Carleman operators from E into F and by $B(E,F)$ the space of all bounded operators. Moreover we put $\ell_\infty B(E,F)=\{(T_n)_n; T_n\in B(E,F), \|(T_n)_n\|=\sup_n\|T_n\|<\infty\}$ and $\ell_\infty C(E,F)=\{(T_n)_n;\ T_n\in C(E,F),\ \sup_{\substack{\|e\|\le 1\\ n\in\mathbb{N}}} |T_n e|\in L_0(X,\mu)\}$.

<u>Theorem 5</u> Let E be a Banach space and F as above. Then the following are equivalent:

1) F is isomorphic to an almost (AM)-space.

2) $\ell_\infty B(E,F)=\ell_\infty C(E,F)$.

<u>Proof</u> 1) $\Longrightarrow$ 2) is easy.

2) $\Longrightarrow$ 1) Let

$$f(t)=\sup_{H\subseteq\mathbb{N}}\frac{\left(\sum_{i\in H}|x_i(t)|^2\right)^{1/2}}{\sup_{\|x^*\|\le 1}\left(\sum_{i\in H}|x^*(x_i)|^2\right)^{1/2}},\quad t\in X.$$

We can extract a sequence $(H_n)_n$ of finite subsets of $\mathbb{N}$ such that $f(t)$ is the countable supremum of sums over H_n. By Theorem 2 we should show that $f(t)<\infty$. For each $\varepsilon>0$, $n\in\mathbb{N}$, by Dvoretzki's theorem there exist a linearly independent system (f_i), $i\le 2^{k_n}$, in E^* (here $k_n=\operatorname{card} H_n$), $(1+\varepsilon)$-isomorphic to the canonical basis of $\ell_2(2^{k_n})$. Let $(r_i^n)\ i\le k_n$, the Rademacher system over (f_i).

Define, now, the operators $T_n(e)=\sum_{i\in H_n} r_i^n(e)x_i$, $n\in\mathbb{N}$. It can be shown that $(T_n)_n\in\ell_\infty B(E,F)\subset\ell_\infty C(E,F)$ and let $g:=\sup_{\substack{\|e\|\le 1\\ n\in\mathbb{N}}}|T_n e|\in L_0(X,\mu)$. Then $(1+\varepsilon)^{-1}\sup_n\left(\sum_{i\in H_n}|x_i(t)|^2\right)^{1/2}\le g(t)$ μ-a.e. Thus $f(t)<\infty$ μ-a.e. ∎

<u>Definition 6</u> Let E be a perfect Banach lattice and let F be a Banach space. $T:E\longrightarrow F$ is called an <u>almost cone absolutely summing operator</u> if there exists a maximal orthogonal system $(e_i)_{i\in I}$ in F such that TP_{e_i} are cone absolutely summing operators for all $i\in I$, i.e. TP_{e_i} map positive summable sequences into absolutely summable one. The linear space of all almost cone absolutely summing operators (abbreviated a.c.a.s.) will be denoted by $ACAS(E,F)$.

<u>Theorem 7</u> Let E and F be two perfect Banach lattices and

let $T:E\longrightarrow F$ be an (o)-continuous operator. Then the following are equivalent:

1) T is an a.c.a.s. operator.

2) $T^x:F^x\longrightarrow E^x$ is a Carleman operator.

Assume now that E is a maximal Köthe function space on the finite measure space (Y,ν). Then anyone of 1) or 2) is equivalent with:

3) $\exists$ $f\in L_0(Y,\nu)$ such that for every $x_i\in E_+$, $i=1,2,\dots,n$ with $\int_Y x_i(t)f(t)d\nu(t)<\infty$, we have

$$\sum_{i=1}^{n}\|Tx_i\|\le\sum_{i=1}^{n}\int_Y x_i(t)f(t)d\nu(t).$$

Proof 1) $\Rightarrow$ 2) It is sufficient to prove the implication whenever F is a maximal Köthe function space. Then there exists a disjoint partition $(A_i)_i$ of Y such that TP_{A_i} are cone absolutely summing operators. But then $P_{A_i}T^x=(TP_{A_i})^x$ are majorizing operators, consequently T^x is a Carleman operator.

2)$\Rightarrow$1) is similar to the previous implication since the majorizing and c.a.s. operators are dual each other.

2)$\Rightarrow$3) Let $f\in L_0(Y,\nu)$ such that $|T^xy^x|\le\|y^x\|_{F^x}\cdot f$ (1).

Let now $x_i\in E_+$ as in the statement. Then $\sum_{i=1}^{n}\|Tx_i\|=$

$$=\sum_{i=1}^{n}\sup_{\|y^x\|\le 1}\int_Y(T^xy^x)x_i d\nu\le(\text{by (1)})\le\sum_{i=1}^{n}\int_Y f\cdot x_i d\nu .$$

3)$\Rightarrow$1) By Luzin's theorem there exists a disjoint partition $(A_n)_n$ of Y such that $\chi_{A_n}\cdot f\in E^x$ for all $n\in\mathbb{N}$. Thus, by 3),

$$\sum_{i=1}^{n}\|TP_{A_k}x_i\|\le\sum_{i=1}^{n}\int_Y x_i\chi_{A_k}fd\nu\le\|\chi_{A_k}f\|_{E^x}\cdot\|\sum_{i=1}^{n}x_i\|_E \quad (\forall)\ k,n\in\mathbb{N}.$$

Hence TP_{A_k} are c.a.s. operators for all $k\in\mathbb{N}$. ∎

Definition 8 A perfect Banach lattice E is an almost (AL)-space (abbreviated a(AL)-space) if Id_E is an a.c.a.s. operator.

Every (AL)-space and E^x, where E is defined after Definition 3, are a(AL)-spaces. Also a perfect M-tensor product of two a(AL)-spaces is an a(AL)-space again.

Let's state the following duality theorem:

Theorem 9 Let E be a perfect Banach lattice. The following are equivalent:

1) E is an a(AL)-space (resp. E is an a(AM)-space).

2) E^x is an a(AM)-space (resp. E^x is an a(AL)-space).

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