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ONE DIAGONAL INTERPOLATION THEOREM AND ITS APPLICATION

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Notations. We shall consider function defined on the interval $[0,1]$. $C^{(k)}$ will be the space of all functions with k-th derivative continuous in $[0,1]$. We set $f^{(k)}\in L_p$, $p \geqq 1$ if $f^{(k-1)}$ is absolutely continuous and is an integral of $f^{(k)}\in L_p$.
We set $W_p^r=\{f : f^{(r)}\in L_p\}$;

$$\|f\|_{\overset{\circ}{W}{}_p^r} = \|f^{(r)}\|_{L_p} ;$$

$$\|g\|_{L_p[0,1]} = \|g\|_p = \left(\int_0^1 (g(x))^p dx\right)^{1/p} ; \; L_\infty = C = C^0 .$$

We call the operator T semi-linear if

$$\|T(f+g)\|_A \leqq \|Tf\|_A + \|Tg\|_A ; \; T: B \to A \text{ , } A,B -$$

Banach spaces.

Let H_n be the set of all algebraic polynomials of n-th degree, $S_{n,k}$ - the set of all splines of k-th degree with n free knots in $[0,1]$, R_n - the set of all rational functions of n-th degree. The corresponding best approximations in L_p are:

$$E_n(f)_{L_p} = \inf\{\|f-p\|_{L_p} : p\in H_n\}$$

$$E_{n,k}(f)_{L_p} = \inf\{\|f-s\|_{L_p} : s\in S_{n,k}\}$$

$$R_n(f)_{L_p} = \inf\{\|f-r\|_{L_p} : r\in R_n\} .$$

1.First of all we should explain what we understand by "diagonal interpolation theorems". The well known Riesz - Thorin interpolation theorem states (in one if its weak versions, see for example [1] , [2] that if we have a linear operator T and T maps L_{p_i} in L_{q_i} , $i = 0,1$ and

$$\|Tf\|_{L_{q_i}} \leqq M_i \|f\|_{L_{p_i}} , \; i = 0,1$$

then T maps L_p in L_q for $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$
with

$$\|Tf\|_{L_q} \leqq M_0^{1-\theta} M_1^{\theta} \|f\|_p , \quad 0 < \theta < 1 .$$

We call this type of theorems (between two L_{p_i} , $i = 0,1$) vertical interpolation theorems.

On the other side there exists another type of theorems - when we consider differentiable functions in a space L_p . One of these theorems, which is easy to obtain using the K- functional of J . Peetre, see [4] , [5] , and is a typical theorem in the theory of approximation, is the following:

If T is a semi-linear operator such that if $f \in C^{(k)}$ than

$$\|Tf\|_C \leqq M_0 \|f\|_C$$

$$\|Tf\|_C \leqq M_k \|f^{(k)}\|_C$$

, then

if $f \in C^{(r)}$, $0 < r < k$, we have

$$\|Tf\|_C \leqq M_0^{1-r/k} M_k^{r/k} \|f^{(r)}\|_C$$

We call such type of theorems "horisontal" interpolation theorems.

Under "diagonal" interpolation theorems we understand theorems of the type:

If T is a (semi) linear operator and

$$\|Tf\|_{A_0} \leqq M_0 \|f\|_{L_{p_0}} , \quad 1 \leqq p_0 \leqq \infty$$

$$\|Tf\|_{A_1} \leqq M_k \|f^{(k)}\|_{L_{p_1}} , \quad 1 \leqq p_1 \leqq \infty , \quad p_1 < p_0 ,$$

then $$\|Tf\|_{A_\theta} \leqq c\, M_0^{1-\theta} M_1^{\theta} \|f^{(r)}\|_{L_p}$$

where c is an absolute constant, $\theta = r/k$,

$1/p = (1-\theta)/p_0 + \theta/p_1$, A_0 , A_1 , A_θ - Banach spaces, (A_0 , A_1 , A_θ can be equal).

Certainly some "diagonal" theorems are well-known. Let us remember only the Calderon's theorem [6] :

(1) $$(W_{p_0}^{s_0}, W_{p_1}^{s_1})_{[\theta]} = W_p^s , \ 1 < p_0, p_1 < \infty ,$$

$$s = (1-\theta) s_0 + \theta s_1 , \ 1/p = (1-\theta)/p_0 + \theta/p_1 , \ 0 < \theta < 1 ,$$

$(A,B)_{[\theta]}$ denotes the interpolation space between A and B obtained by means of the complex method of interpolation using parameter A (see [2] , [3]).

Let us remark that here $1 < p_0 < \infty$, $1 < p_1 < \infty$ and this restriction appears in the proof because of Michlin's theorem for multipliers.

But the case $p_0 = \infty$, $p_1 \geq 1$ is one of the most interesting cases, at least in approximation theory (I suppose also in applications in the theory and numerical methods for differential equations). For example from the estimations

(2) $$E_n(f)_p \leqq M_0 \| f \|_p$$
$$E_n(f)_p \leqq M_k \| f^{(k)} \|_p$$

it follows

(3) $$E_n(f)_p \leqq M_0^{1-r/k} M_k^{r/k} \| f^{(r)} \|_p ,$$

$1 \leqq p \leqq \infty$, $0 < r < k$, $E_n(f)_p$ - an operator of the type of best approximation in L_p .

The estimation (3) is a typical "horizontal" interpolation theorem in the theory of approximation. Usual $M_0 = 1$, $M_k = c(k) n^{-k}$ and then (3) gives

$$E_n(f)_p \leqq c(r,k) n^{-r} \| f^{(r)} \|_p$$

$c(r,k)$ - a constant, depending only on r and k.

Estimations of the type (2) are typical for the classical polynomial approximations (with algebraic or triginometrical polynomials). But recently the theory of spline and rational evolved in a considerable extent. It is known that the right estimations in these theories are not of the type (2).

For example for best approximation by polygons with free knots we have [7] :

$$E_{n,1}(f)_C \leqq \|f\|_C \; ; \; E_{n,1}(f)_C \leqq c' n^{-1} \|f'\|_{L_1} ,$$

where c' is an absolute constant.

For rational approximations we have the following estimations:

(4) $$R_n(f)_C \leqq \|f\|_C$$ (the trivial estimation)

(5) $$R_n(f)_C \leqq c(p)\, n^{-1} \|f'\|_p \;, \; p > 1 ,$$

$c(p)$ constant, depending only on p (see Y.A.Brudnii [10] , V.A.Popov [9]),

(6) $$R_n(f)_C \leqq c'' n^{-2} \|f''\|_p \;, \; p \geqq 1 ,$$

c'' - an absolute constant [8] .

The proofs of (5) and (6) are similar, but different - from the proof of (6) it does not follows the proof of (5).

My interest in "diagonal interpolation theorems" came with the question: if (5) (at least for $p \geq 2$) follows from (6) by means of general considerations, not by using actually rational function's arguments.

The answer is positive (for (5) with $p \geqq 2$) and it follows from the results below.

2, We shall need the following lemma:

<u>Lemma 1</u>. Let the function f be such that $f' \in L_p$, $p \geqq 1$. Then for every $\varepsilon > 0$ there exists a function f_ε such that

$$\|f - f_\varepsilon\|_C \leqq \varepsilon \|f'\|_{L_p}$$
$$\|f_\varepsilon^{(k)}\|_{L_q} \leqq c(k)\, \varepsilon^{1-k} \|f'\|_{L_p} ,$$

where $q = p/k$ and the constant $c(k)$ depends only on k.

Proof. Let $\varepsilon > 0$ be given. Obviously we may assume that $\|f'\|_p = 1$. Let us divide the interval [0,1] into m parts by means the points $0 = x_0 < x_1 < \ldots < x_m = 1$ constructed by induction as follows:

Let $x_0 = 0$. Then x_1 is the first $x > x_0$ such that $|f(x_1) - f(x_0)| = \varepsilon$ .Then $|f(x) - f(x_0)| < \varepsilon$ for $x \in [x_0, x_1]$. If such x_1 does not exist, we set $x_1 = 1$. Let x_i be given. Then x_{i+1} is the first $x > x_i$ such that $|f(x_{i+1}) - f(x_i)| = \varepsilon$. Then $|f(x) - f(x_i)| < \varepsilon$ for $x \in [x_i, x_{i+1}]$. If such x_{i+1} does not exist, we set $x_{i+1} = 1$. Since

$f' \in L_p$, $p \geqq 1$, f is absolutely continuous (see Notations) and after finite number of steps, m steps, we shall have $x_m = 1$. So we have points x_i , $i = 0, \ldots, m$, $0 = x_0 < x_1 < \ldots < x_m = 1$;
with the properties:

(7)
$$\begin{aligned} &|f(x_i) - f(x_{i-1})| = \varepsilon \ , \ i = 1, \ldots, m-1 \ ; \\ &|f(x) - f(x_i)| \leqq \varepsilon \ , \ i = 0, \ldots, m-1 \ ; \\ &|f(x_m) - f(x_{m-1})| \leq \varepsilon \end{aligned}$$

Let us estimate now $\sum_{i=1}^{m-1} h_i^{1-p}$, $p \geqq 1$, $h_i = x_i - x_{i-1}$, $i = 0, \ldots, m-1$.

We have $(1/p + 1/q = 1)$,using Holder's inequality

$$\varepsilon = |f(x_i) - f(x_{i-1})| = \left|\int_{x_{i-1}}^{x_i} f'(t)\,dt\right|$$

$$\leqq (x_i - x_{i-1})^{1/q} \left(\int_{x_{i-1}}^{x_i} |f'(t)|^p\,dt\right)^{1/p} ,$$

or
$$\varepsilon^p h_i^{-p/q} \leqq \int_{x_{i-1}}^{x_i} |f'(t)|^p\,dt$$

From here we obtain:

(8)
$$\sum_{i=1}^{m-1} \varepsilon^p h_i^{-p/q} \leqq \int_0^{x_{m-1}} |f'(t)|^p\,dt \leq \|f'\|_p^p = 1$$

(Since we have assumed that $\|f'\|_p = 1$)

Since $-p/q = -(1-1/p) = 1-p$ we obtain from (8):

$$\varepsilon^p \sum_{i=1}^{m-1} h_i^{1-p} \leqq 1 , \quad \text{or}$$

(9)
$$\sum_{i=1}^{m-1} h_i^{1-p} \leqq \varepsilon^{-p} .$$

Let us construct now the function f_ε . From the theory of the spline functions it is well-known that for every points x_{i-1} , x_i , $x_{i-1} < x_i$,there exists a spline of k-th degree $s_k(x_{i-1}, x_i ; x)$ with equidistant knots in the interval $[x_{i-1}, x_i]$ with the properties:

$$s_k(x_{i-1}, x_i; x) = 0 \quad \text{for} \quad x \leq x_{i-1}$$

(10)

$$s_k(x_{i-1}, x_i; x) = 1 \quad \text{for} \quad x \geq x_i$$

$0 \leq s_k(x) \leq 1$; $|s_k^{(k)}(x_{i-1}, x_i; x)| \leq c(k)\, h_i^{-k}$ for all $x \in (x_{i-1}, x_i)$ except the knots of the spline (I.J.Schoenberg [11] , see also [12]).

Now we set:

$$f_\varepsilon(x) = f(x_0) + \sum_{i=1}^{m-1} (f(x_i) - f(x_{i-1}))\, s_k(x_{i-1}, x_i; x)$$

From (10) it follows that $f_\varepsilon(x_i) = f(x_i)$, $i = 0, \ldots, m-1$,

$f(x) = f(x_{m-1})$ for $x_{m-1} \leq x \leq x_m = 1$.

On the other hand from the construction of the points x_i , (7) and (10) it follows, that we have

(11) $$\| f - f_\varepsilon \|_C \leq \varepsilon$$

and for all $x \in (x_{i-1}, x_i)$, except the knots of the spline $s_k(x_{i-1}, x_i; x)$ we have

(12) $$| f_\varepsilon^{(k)}(x) | \leq \varepsilon\, c(k)\, h_i^{-k}$$

Let us calculate now $\| f_\varepsilon^{(k)} \|_{L_q}$. Using (12) we obtain:

$$\| f_\varepsilon^{(k)} \|_q = \left(\int_0^1 | f_\varepsilon^{(k)}(x) |^q\, dx \right)^{1/q} \leq \left(\sum_{i=1}^{m-1} \left(\frac{c(k)\varepsilon}{h_i^k} \right)^q h_i \right)^{1/q}$$

$$= c(k)\, \varepsilon \left(\sum_{i=1}^{m-1} 1/h_i^{kq-1} \right)^{1/q}$$

If we want to use the estimation (9) we must set $q = p/k$ and so we obtain:

$$\| f_\varepsilon^{(k)} \|_{p/k} \leq c(k)\, \varepsilon^{1-p/q} = c(k)\, \varepsilon^{1-k}$$

The lemma 1 is proved.

Let us remark, that in the calculation of $\| f_\varepsilon^{(k)} \|_q$ we have not use that $\| f_\varepsilon^{(k)} \|_q$ is a norm, so we have lemma 1 also for $q = p/k < 1$.

3. Let we consider the K-functional of J.Peetre between C and $\mathring{W}_q^k$ (see [4] , [5]);

$$K(C,\mathring{W}_q^k ; f ; t) = \inf_{f=f_1+f_2} \{ \|f_1\|_C + t\|f^{(k)}\|_q \}$$

<u>Theorem 1</u>. Let f be such that $f' \in L_p$, $p \geqq 1$

Then

$$K(C,\mathring{W}_q^k ; f,t) \leqq c'(k) t^{1/k} \|f'\|_p , \quad q = p/k ,$$

where $c'(k)$ is a constant depending only on k.

Proof. Using lemma 1 we have for $q = p/k$:

$$K(C,\mathring{W}_q^k ; f,t) \leqq \|f-f_\varepsilon\|_C + t\|f_\varepsilon^{(k)}\|_q$$

$$\leqq \varepsilon \|f'\|_p + t\, c(k)\, \varepsilon^{1-k} \|f'\|_p .$$

If we set $\varepsilon = t^{1/k}$ we obtain

$$K(C,\mathring{W}_q^k ; f,t) \leqq (1+c(k)) t^{1/k} \|f'\|_p$$

<u>Theorem 2</u>. Let T be a (semi) linear operator such that

$$\|Tf\|_A \leqq M_0 \|f\|_C$$

$$\|Tf\|_A \leqq M_k \|f^{(k)}\|_q$$

where A is a Banach space (usually A=Cor L_p).

If $f' \in L_p$ then

$$\|Tf\|_A \leqq c''(k) M_0^{1-1/k} M_k^{1/k} \|f'\|_p , \quad p = kq ,$$

where $c''(k)$ is a constant depending only on k.

Proof. We have, using lemma 1:

$$\|Tf\|_A \leqq \|T(f-f_\varepsilon)\|_A + \|Tf_\varepsilon\|_A$$

$$\leqq M_0 \varepsilon \|f'\|_p + M_k\, c(k)\, \varepsilon^{1-k} \|f'\|_p$$

If we set $\varepsilon = M_0^{-1/k} M_k^{1/k}$ we obtain the statement of the theorem.

Now we shall show how (5) for $p \geqq 2$ follows from (6) We have, using lemma 1, if $f' \in L_p$; $k=2$:

$$R_n(f)_C \leqq \|f-f_\varepsilon\|_C + R_n(f_\varepsilon)_C \leqq \varepsilon\|f'\|_p + c'' n^{-2} \|f_\varepsilon''\|_{p/2}$$

$$\leqq \varepsilon\|f'\|_p + c'' n^{-2} c(k)\, \varepsilon^{-1} \|f'\|_p .$$

If we set $\varepsilon = n^{-1}$, we obtain (5) for $p \geqq 2$

REMARKS. 1.Using an analogue method it is possible to obtain that if $f^{(r)} \in L_p$, $p \geqq 1$, then

$$K(C, \overset{\circ}{W}{}_q^k; f, t) = O(t^{r/k}),\ q = pr/k \ .$$

2. A natural question what happens if we consider fractional derivatives we shall consider later.

3. In theorems 1,2 q can be <1 . But we don't know some operators of the type of best approximation for which we have an estimation of the type $\|Tf\|_A \leqq M_k \|f^{(k)}\|_q$, $q<1$. Usually for $p<1$ we must use another norms (for example Besov's norms, see [10]).

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