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TRIGONOMETRIC INTERPOLATION OF FUNCTIONS OF BOUNDED VARIATION

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Let BV denote the set of all 2π-periodic complex-valued functions of bounded variation. We consider for $f\in BV$ the trigonometric interpolatory polynomial of degree n

$$(L_nf)(x) = \frac{2}{2n+1}\sum_{k=0}^{2n} f(x_k)\, K_n(x-x_k)$$

with the Dirichlet kernel

$$K_n(x) = \frac{1}{2} + \sum_{m=1}^{n} \cos mx$$

based on the equidistant nodes

$$x_k = \frac{2k\pi}{2n+1} \qquad (k=0,1,\dots,2n).$$

The purpose of this note is to give the exact order of L^p-convergence of L_nf to $f\in BV$ for $1\le p<\infty$. A first result in this direction was stated by S. Prößdorf, B. Silbermann [1] and K. Zacharias [2] in the case p=2. In [3] we proved with help of estimations of the Fourier-coefficients the following result for $p\ge 2$:

$$\|f - L_nf\|_p = O(n^{-l-1/p}) \qquad n\to\infty,$$

if the l-th derivative of f exists in BV. It is interesting to compare these results with the best approximation to f and the approximation by the n-th Fourier-sum S_nf. We have for $f\in BV$ the well-known estimations

$$E_n(f,L^p) \le c\ V(f)\ n^{-1/p} \qquad (1\le p<\infty)$$

and

$$\|f - S_n f\|_p \leq V(f)\, n^{-1/p} \cdot \begin{cases} A \ln n & \text{if } p=1, \\ A(p) & \text{if } 1<p<\infty, \end{cases}$$

which are in general the best possible.
Now we formulate the analogous result for interpolation.

<u>Theorem.</u> If $f \in BV$ and $n=1,2,\ldots$, then

$$\|f - L_n f\|_p \leq B \cdot V(f) \cdot (2n+1)^{-1/p} \cdot \begin{cases} 22 + 4.2 \ln n & \text{if } p=1, \\ C(p) & \text{if } 1<p<\infty, \end{cases} \tag{1}$$

where

$$B = \begin{cases} 1 & \text{if } f \text{ real-valued}, \\ \sqrt{2} & \text{if } f \text{ complex-valued}. \end{cases}$$

Further $C(p)$ depends only on p and

$$C(p) \leq \begin{cases} 18 + 6/(p-1) & \text{if } 1<p<2, \\ 10 & \text{if } p=2, \\ 20 & \text{if } 2<p<\infty. \end{cases}$$

For the proof we need some Lemmas.

<u>Lemma 1.</u> For $n \geq 1$ and $x_k \leq x \leq x_{k+1}$ we have

$$|K_n(x)| \leq \begin{cases} (2n+1)/2 & \text{if } k=0,\ k=2n, \\ (2n+1)/4k & \text{if } 0<k\leq n, \\ (2n+1)/4(2n-k) & \text{if } n<k<2n, \end{cases}$$

and

$$|K_n(x) + K_n(x+x_1)| \leq \begin{cases} (2n+1)/2 & \text{if } k=0,\ k=2n-1, \\ \pi(2n+1)/8k^2 & \text{if } 0<k\leq n-1, \\ \pi(2n+1)/8(2n-k-1)^2 & \text{if } n\leq k<2n-1. \end{cases}$$

The proof follows directly from the representation of the Dirichlet kernel.

<u>Lemma 2.</u> For $0 \leq j < 2n$ we have

$$\int_{x_{j+1}}^{2\pi} \Big|\sum_{k=0}^{j} K_n(x-x_k)\Big|^p dx + \int_0^{x_j} \Big|\sum_{k=j+1}^{2n} K_n(x-x_k)\Big|^p dx$$

$$\leq \begin{cases} 56.1 + 13.1 \ln n & \text{if } p=1, \\ c(p)\cdot(2n+1)^{p-1} & \text{if } 1<p<\infty, \end{cases}$$

with

$$c(p) < 2\pi \cdot \begin{cases} (7+3/(p-1))^p & \text{if } 1<p<2, \\ 20 & \text{if } p=2, \\ 8^p & \text{if } 2<p<\infty . \end{cases}$$

Proof. Let

$$A_r = \int_{x_r}^{x_{r+1}} \left| \sum_{k=o}^{j} K_n(x-x_k) \right|^p dx \qquad (j<r<2n+1),$$

and

$$B_s = \int_{x_s}^{x_{s+1}} \left| \sum_{k=j+1}^{2n} K_n(x-x_k) \right|^p dx \qquad (0\le s<j).$$

Then we obtain

$$A_r \le \frac{2\pi}{2n+1} \max_{x_r\le x\le x_{r+1}} \left| \sum_{k=o}^{j} K_n(x-x_k) \right|^p .$$

For even j we define $d = \min(r-j, 2n-r+1)$, $e = \min(2n-r, r)$ and $f = \min(d,e) = \min(r-j, 2n-r)$. We get with Lemma 1 for $r<2n$

$$A_r < \frac{2\pi}{2n+1} \left(\frac{\pi}{8}(2n+1)\cdot 2 \sum_{i=o}^{\infty} (d+2i)^{-2} + \frac{2n+1}{4e}\right)^p$$

and

$$A_{2n} < \frac{2\pi}{2n+1} \left(\frac{\pi}{8}(2n+1)\cdot 2 \sum_{i=o}^{\infty} (d+2i)^{-2} + \frac{2n+1}{2}\right)^p .$$

It follows by summation that

$$\sum_{r=j+1}^{2n} A_r < \frac{2\pi}{2n+1}\left(\frac{2n+1}{4}\right)^p \left\{ \left(\frac{3\pi}{2}+2\right)^p + 2\sum_{f=1}^{n} \left(\frac{(1+\pi/2)f + \pi}{f^2}\right)^p \right\} .$$

The estimation of ΣB_s and the case of odd j can be handled as above. We get the constants by easy calculations.■

Lemma 3. For $0\le j\le 2n$ we have

$$\int_{x_j}^{x_{j+1}} \left\{ \left| \sum_{k=o}^{j} K_n(x-x_k) \right| + \left| \sum_{k=j+1}^{2n} K_n(x-x_k) \right| \right\}^p dx \le \frac{2\pi}{2n+1} \cdot 2^p \cdot (2n+1)^p .$$

Proof. An estimation with the help of Lemma 1 yields

$$\max_{x_j\le x\le x_{j+1}} \left\{ \left| \sum_{k=o}^{j} K_n(x-x_k) \right| + \left| \sum_{k=j+1}^{2n} K_n(x-x_k) \right| \right\}^p \le$$

$$\leq \left\{2n+1 + \frac{\pi}{8}(2n+1)\sum_{i=1}^{\infty} i^{-2}\right\}^p < (2n+1)^p \cdot 2^p \quad . \blacksquare$$

Proof of the Theorem. We set $f=g-h$ in $[0,2\pi)$ for given real-valued f, where g and h are monotone increasing. Further we assume w.l.o.g. $g(0) = h(0) = 0$. For $0 \leq j \leq 2n$ we define

$$g_j(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq x_j \ , \\ g(x) - g(x_j) & \text{if } x_j < x < x_{j+1} \ , \\ g(x_{j+1}) - g(x_j) & \text{if } x_{j+1} \leq x < 2\pi \ . \end{cases}$$

Then we have

$$g = \sum_{j=o}^{2n} g_j \ , \qquad V(g) = \sum_{j=o}^{2n} V(g_j) \ , \tag{2}$$

and

$$\|g - L_n g\|_p \leq \sum_{j=o}^{2n} \|g_j - L_n g_j\|_p \ . \tag{3}$$

Now we consider

$$\|g_j - L_n g_j\|_p = \frac{2}{2n+1}\left\{\frac{1}{2\pi}\int_o^{2\pi}\left|\sum_{k=o}^{2n}(g_j(x)-g_j(x_k))K_n(x-x_k)\right|^p dx\right\}^{1/p} =$$

$$= \frac{2}{2n+1}\left\{\frac{1}{2\pi}\left(\int_o^{x_j}\left|\sum_{k=j+1}^{2n} g_j(x_k)K_n(x-x_k)\right|^p dx + \int_{x_{j+1}}^{2\pi}\left|\sum_{k=o}^{j} g_j(x)K_n(x-x_k)\right|^p dx\right.\right.$$

$$\left.\left. + \int_{x_j}^{x_{j+1}}\left|\sum_{k=o}^{2n}(g_j(x)-g_j(x_k))K_n(x-x_k)\right|^p dx\right)\right\}^{1/p} \ .$$

By Lemma 2 and Lemma 3 follows in the case $p>1$

$$\|g_j - L_n g_j\|_p < (2n+1)^{-1/p}\, V(g_j)\cdot 2\cdot(2^p+c(p)/2\pi)^{1/p} \ .$$

For $p=1$ we obtain analogously

$$\|g_j - L_n g_j\|_1 < V(g_j)\cdot(22+4.2 \ln n)/(2n+1) \ .$$

The Theorem follows now by (2), (3) and

$$\|f - L_n f\|_p \leq \|g - L_n g\|_p + \|h - L_n h\|_p \ ,$$

$$V(f) = V(g) + V(h) \ .$$

Finally let us consider complex-valued f. Setting $f = f_1 + if_2$, we get the assertion with

$$V(f_1) + V(f_2) \le \sqrt{2}\, V(f) . \blacksquare$$

Remark. The rate of convergence in (1) can not be improved in general. To show this, we give a simple example of a function $f \in BV$

$$f(x) = \begin{cases} 1 & \text{if } x=0, \\ 0 & \text{if } 0<x<2\pi . \end{cases}$$

A calculation yields

$$\|f - L_n f\|_p = \frac{2}{2n+1}\Big(\frac{1}{2\pi}\int_0^{2\pi} |K_n(x)|^p \, dx\Big)^{1/p} \ge \begin{cases} A\, n^{-1} \ln n & \text{if } p=1, \\ B\, n^{-1/p} & \text{if } 1<p<\infty. \end{cases}$$

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