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UNIFORMLY DISTRIBUTED MATRICES AND NUMERICAL INTEGRATION

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Let $X = (x_k^{(n)})$, $n = 1,2,\dots$; $k = 1,2,\dots,n$, be a given infinite triangular matrix of points belonging to the interval $E = [0,1]$ and $P = (p_k^{(n)})$, $n = 1,2,\dots$; $k = 1,2,\dots,n$, be a given infinite triangular matrix of nonnegative numbers. We call the matrix P weight of the matrix X.

The matrix X is said to be uniformly distributed with weight P if for every interval $J \subset E$

$$\lim_{n\to\infty} \sum_{k=1}^{n} p_k^{(n)} C(J; x_k^{(n)}) = |J| ,$$

where $C(J;x)$ is the characteristic function of the interval J and $|J|$ is its length.

For each natural number n we set

$$D_n^{(q)}(X,P) = \left(\int_0^1 | G_n(X,P;x) |^q dx \right)^{1/q} , \qquad 0 < q \le \infty ,$$

where

$$G_n(X,P;x) = x - \sum_{k=1}^{n} p_k^{(n)} C([0,x); x_k^{(n)}) .$$

Instead of $D_n^{(\infty)}(X,P)$ we simply write $D_n(X,P)$.

It is easy to prove that if the matrix X is uniformly distributed with weight P then for every q $(0 < q \le \infty)$

(1) $$\lim_{n\to\infty} D_n^{(q)}(X,P) = 0 .$$

It may be proved also that if equality (1) holds for certain q with $0 < q \leq \infty$ then the matrix X is uniformly distributed with weight P.

Now let us consider the general linear process of the numerical integration:

$$I(f) = \int_0^1 f(x)dx \approx Q_n(f;X,P) = \sum_{k=1}^{n} p_k^{(n)} f(x_k^{(n)})$$

whose error for the function f is defined as the quantity

$$R_n(f;X,P) = I(f) - Q_n(f;X,P), \qquad n = 1,2,\ldots$$

In [1] we proved that

$$\lim_{n\to\infty} R_n(f;X,P) = 0$$

holds for every Riemann integrable function f on E if and only if the matrix X is uniformly distributed with weight P. In [1] we proved also that if the matrix P satisfies the following condition

(2) $$p_1^{(n)} + p_2^{(n)} + \ldots + p_n^{(n)} = 1$$

then for every continuous function f on E

$$|R_n(f;X,P)| \leq \omega(f;D_n(X,P)),$$

where $\omega(f)$ is the modulus of continuity of the function f.

In [2] we proved that if the matrix P satisfies (2) then for every Riemann integrable function f on E

$$|R_n(f;X,P)| \leq \tau(f;2D_n(X,P)),$$

where $\tau(f)$ is the averaged modulus of smoothness of f. We recall that

$$\tau(f,t) = \int_0^1 \omega(f,x;t)dx, \qquad t>0,$$

where

$$\omega(f,x;t) = \sup\{ |f(y) - f(z)| : y,z \in [x-t/2,x+t/2] \cap E \}.$$

For the history of the modulus $\tau(f)$ and its properties see [3].

Let us denote by $H_q(C)$, $0<q\leq 1$, the class of functions f defined on E for which the inequality

(3) $$|f(x) - f(y)| \leq C|x - y|^q$$

holds true for every x and y belonging to the interval E ;here C is an absolute constant.

Sobol ([4] , p. 65) proved that if the matrix P satisfies (2) then for every function $f \in H_1(C)$

(4) $$|R_n(f;X,P)| \leq C\, D_n^{(1)}(X,P)$$

and the estimate (4) is the best possible.

In this paper we generalize the estimate (4) for the classes $H_q(C)$, $0<q\leq 1$, namely we prove the following

Theorem 1. Let the matrix P satisfy (2). Then for every function $f \in H_q(C)$, $0<q\leq 1$, we have

(5) $$|R_n(f;X,P)| \leq C(D_n^{(q)}(X,P))^q .$$

The proof of Theorem 1 follows from the following

Lemma 1. Let the matrix P satisfy the following condition

(6) $$x_1^{(n)} \leq x_2^{(n)} \leq \dots \leq x_n^{(n)}$$

and the matrix P satisfy (2). Then for $0<q<\infty$ we have

(7) $$D_n^{(q)}(X,P) = \left(\sum_{k=1}^{n} \int_{a_{k-1}^{(n)}}^{a_k^{(n)}} |x - x_k^{(n)}|^q dx \right)^{1/q} ,$$

where

(8) $$a_0^{(n)} = 0 , \quad a_k^{(n)} = \sum_{i=1}^{k} p_i^{(n)} \quad (k=1,2,\dots,n) .$$

Proof. For notational convenience, we set $x_0^{(n)} = 0$ and $x_{n+1}^{(n)} = 1$. The distinct values of the numbers $x_i^{(n)}$, $i=0,1,\dots,n+1$, define a subdivision of the interval E . It is easy to see that from the definition of the function $G_n(X,P;x)$ and from (6) follows that if $x \in (x_k^{(n)}, x_{k+1}^{(n)}]$, $0\leq k\leq n$, then

$$G_n(X,P;x) = x - a_k^{(n)}$$

Therefore

$$(9)\qquad (D_n^{(q)}(X,P))^q = \sum_{k=0}^{n} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} |x - a_k^{(n)}|^q dx .$$

Using the equality (see [4], p. 279)

$$(10)\qquad \int_a^b |x-c|^q dx = \frac{1}{q+1}\left(|b-c|^q(b-c) + |c-a|^q(c-a)\right),$$

which holds for all real numbers a, b and c, we obtain from (9)

$$(11)\qquad (D_n^{(q)}(X,P))^q =$$

$$= \frac{1}{q+1}\sum_{k=0}^{n}\left(|x_{k+1}^{(n)} - a_k^{(n)}|^q(x_{k+1}^{(n)} - a_k^{(n)}) + |a_k^{(n)} - x_k^{(n)}|^q(a_k^{(n)} - x_k^{(n)})\right).$$

From (8) and (2) we conclude that $a_k^{(n)} = 1$. But we also have $a_o^{(n)} = 0$, $x_o^{(n)} = 0$ and $x_{n+1}^{(n)} = 1$. Therefore, from (11) we get

$$(12)\qquad (D_n^{(q)}(X,P))^q = \frac{1}{q+1}\sum_{k=o}^{n-1}|x_{k+1}^{(n)} - a_k^{(n)}|^q(x_{k+1}^{(n)} - a_k^{(n)})$$

$$+ \frac{1}{q+1}\sum_{k=1}^{n}|a_k^{(n)} - x_k^{(n)}|^q(a_k^{(n)} - x_k^{(n)})$$

$$= \frac{1}{q+1}\sum_{k=1}^{n}\left(|x_k^{(n)} - a_{k-1}^{(n)}|^q(x_k^{(n)} - a_{k-1}^{(n)}) + |a_k^{(n)} - x_k^{(n)}|^q(a_k^{(n)} - x_k^{(n)})\right).$$

On the other hand, using again the equality (10) we obtain

$$(13)\qquad \sum_{k=1}^{n}\int_{a_{k-1}^{(n)}}^{a_k^{(n)}} |x - x_k^{(n)}|^q dx$$

$$= \frac{1}{q+1}\sum_{k=1}^{n}\left(|a_k^{(n)} - x_k^{(n)}|^q(a_k^{(n)} - x_k^{(n)}) + |x_k^{(n)} - a_{k-1}^{(n)}|^q(x_k^{(n)} - a_{k-1}^{(n)})\right).$$

Now (7) follows from (12) and (13). The lemma is proved.

From Lemma 1 the following assertion for the case $q = \infty$ immediately follows.

Corollary [1]. Let the matrix X satisfy (6) and the matrix P satisfy (2). Then

$$D_n(X,P) = \max\left\{ \left|x_k^{(n)} - a_{k-1}^{(n)}\right|, \left|x_k^{(n)} - a_k^{(n)}\right| : k = 1,2,\dots,n \right\}.$$

Proof of Theorem 1. Let $f \in H_q(C)$, $0 < q \leq 1$. From (2) and (8) we have

$$R_n(f;X,P) = \sum_{k=1}^{n} \int_{a_{k-1}^{(n)}}^{a_k^{(n)}} (f(x) - f(x_k^{(n)}))dx . \tag{14}$$

From (14) and (3) we obtain

$$|R_n(f;X,P)| \leq C \sum_{k=1}^{n} \int_{a_{k-1}^{(n)}}^{a_k^{(n)}} |x - x_k^{(n)}|^q dx . \tag{15}$$

If the matrix X satisfies (6) then the estimate (5) follows from (15) and Lemma 1.

Now let the matrix X does not satisfy (6). Let i_1, i_2, ..., i_n be a permutation of the numbers 1, 2, ..., n. Let us form the matrices X' and P' from the matrices X and P changing their n-th rows with

$$(x_{i_1}^{(n)}, x_{i_2}^{(n)}, \dots, x_{i_n}^{(n)}) \quad \text{and} \quad (p_{i_1}^{(n)}, p_{i_2}^{(n)}, \dots, p_{i_n}^{(n)})$$

respectively. It is easy to see that

$$D_n^{(q)}(X',P') = D_n^{(q)}(X,P) \quad \text{and} \quad R_n(f;X',P') = R_n(f;X,P) . \tag{16}$$

Now let us choose the permutation i_1, i_2, ..., i_n so that

$$x_{i_1}^{(n)} \leq x_{i_2}^{(n)} \leq \dots \leq x_{i_n}^{(n)} . \tag{17}$$

In fact, (17) is the condition (6) for the matrix X'. Therefore, from what we have proved above it follows that

$$|R_n(f;X',P')| \leq C(D_n^{(q)}(X',P'))^q .$$

From here and (16) we again obtain the estimate (5). The theorem is proved.

In conclusion we shall note that in some cases inequality (5) reduces to equality. Here is an example: Let $p_k^{(n)}$ $(k=1,2,\dots,n)$ satisfy (2) and

$$a_k^{(n)} = (x_k^{(n)} + x_{k+1}^{(n)})/2 \qquad (k=1,2,\dots,n-1),$$

where the numbers $a_k^{(n)}$ are defined by (8). Let us define the function f on E by

$$f(x) = C\left|x - x_k^{(n)}\right|^q \quad \text{if} \quad x \in \left[a_{k-1}^{(n)}, a_k^{(n)}\right].$$

Obviously $f \in H_q(C)$. Now from (14) and Lemma 1 we conclude that in this case the equality in (5) holds.

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