КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 710—717

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p.710-717

STRONGLY ELLIPTIC SINGULAR INTEGRAL EQUATIONS WITH PIECEWISE CONTINUOUS COEFFICIENTS AND THE CONVERGENCE OF SPLINE GALERKIN AND COLLOCATION METHODS

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1. Introduction

Let Γ be a closed or open oriented plane Ljapunov curve. By $L^2(\Gamma,\mathbb{C}^m)$ we denote the Hilbert space of all square Lebesgue-integrable $\mathbb{C}^m$-valued functions on Γ with scalar product

$$(f,g) := \int_\Gamma [f(t),g(t)] \; |dt| \;, \; \forall f,g \in L^2(\Gamma,\mathbb{C}^m)$$

Herein $[.,.]$ is the usual scalar product in $\mathbb{C}^m$. $\mathbb{C}^{m\times m}$ stands for the set of all complex-valued $m\times m$ matrices. The symbol $PC(\Gamma,\mathbb{C}^{m\times m})$ designates the space of all $\mathbb{C}^{m\times m}$-valued functions a on Γ which are piecewise continuous in the following sense: for each $t\in\Gamma$ the finite limits $a(t\pm 0) := \lim_{\tau\to t\pm 0} a(\tau)$, $\tau\in\Gamma$ (with respect to the orientation of Γ) exist and a is discontinuous at most at a finite number of points $t\in\Gamma$. $C(\Gamma,\mathbb{C}^{m\times m})$ is the subspace of all continuous $\mathbb{C}^{m\times m}$-valued functions on Γ.

In $L^2(\Gamma,\mathbb{C}^m)$ we consider the singular integral operator of the form

$$A := aP_\Gamma + bQ_\Gamma \tag{1.1}$$

with coefficients $a,b\in PC(\Gamma,\mathbb{C}^{m\times m})$. Here P_Γ and Q_Γ denote the operators

$$P_\Gamma := \frac{1}{2}(I + S_\Gamma),\; Q_\Gamma := \frac{1}{2}(I - S_\Gamma)$$

with the identity operator I and the Cauchy singular operator S_Γ:

$$(S_\Gamma x)(t) := \frac{1}{\pi i} \int_\Gamma \frac{x(\tau)}{\tau - t} d\tau \quad (t \in \Gamma)$$

It is well known that $A \in \mathcal{L}(L^2(\Gamma, \mathbb{C}^m))$. We call the operator A defined by (1.1) <u>strongly elliptic</u>, if there exist a compact operator $T \in \mathcal{L}(L^2(\Gamma, \mathbb{C}^m))$ and an invertible function $\Theta \in PC(\Gamma, \mathbb{C}^{m \times m})$ which is discontinuous at most at the points of discontinuity of a or b such that $A = \Theta(A_0 + T)$, where A_0 has a positive definite real part - i.e.

$$\mathrm{Re}(A_0 f, f) \geq \mathcal{E}(f,f), \ \forall f \in L^2(\Gamma, \mathbb{C}^m)$$

with $\mathcal{E}$ (const) > 0. We remark that the concept of strongly elliptic operators in the afore mentioned sense was introduced by Wendland [15] for the case of continuous functions Θ, a, b. In that case the operator (1.1) is strongly elliptic if and only if $\forall t \in \Gamma$:

$$\det(\mu a(t) + (1-\mu) b(t)) \neq 0, \ \forall \mu \in [0,1] \tag{1.2}$$

(see [11])

In this paper we apply the Galerkin method with polynomial splines and the piecewise linear collocation to the operator (1.1) in case of piecewise continuous coefficients. We obtain that both methods converge in $L^2(\Gamma, \mathbb{C}^m)$ if and (at least in the cases m = 1 or $a, b \in C(\Gamma, \mathbb{C}^{m \times m})$) only if the operator (1.1) is strongly elliptic. In the case of continuous coefficients, analogous results were proved in [4], [10], [11], [14]. Our proofs are quite different from the proofs given there (for complete proofs see the authors' papers [8], [9]).

2. A Sufficient Condition for the Strong Ellipticity

First we shall formulate a condition on the coefficients a and b which implies the strong ellipticity of the operator (1.1).

<u>Theorem 1. The operator (1.1) is strongly elliptic, if the following condition (B) is satisfied:</u>

<u>(B) $\forall t \in \Gamma$: $b(t \pm 0)$ have inverses and there exists $C \in \mathbb{C}^{m \times m}$ such that</u>

$$\mathrm{Re}\, C > 0 \quad \text{and} \quad \mathrm{Re}\, C(b^{-1}a)(t \pm 0) > 0.$$

(If $\Gamma = (\alpha, \beta)$ is an open curve, then condition (B) need only be satisfied from the right or from the left, according as we consider α or β respectively.)

The proof of Theorem 1 is an algebraic one combined with some localization techniques (see [9] ,[7]).

For two important cases, the subsequent lemma gives conditions which are equivalent to (B) (compare with [8] and [11]).

Lemma 1. (a) If $m = 1$, then (B) holds at $t \in \Gamma$ if and only if $b(t\pm0) \neq 0$ and

$$\forall \mu \in [0,1] : \mu b^{-1}a(t+0) + (1-\mu)b^{-1}a(t-0) \notin (-\infty,0]$$

(b) If a and b are continuous at $t \in \Gamma$, then condition (B) holds at t if and only if (1.2) is satisfied.
(c) In the cases (a) and (b) condition (B) is equivalent to the strong ellipticity condition.

3. Strong Ellipticity and the Convergence of the Galerkin Method

Now we consider the operator $A = aP_\Gamma + bQ_\Gamma \in \mathcal{L}(L^2(\Gamma,\mathbb{C}^m))$ with $a,b \in PC(\Gamma,\mathbb{C}^{m\times m})$ and suppose A to be invertible. Since Γ is a simple Ljapunov curve, it is given by a regular parametrization

$$\Gamma := \{ t = \gamma(s) \mid s \in [0,1] \}$$
$$\gamma : [0,1] \to \mathbb{C}$$

For closed curves we assume $\gamma(0) = \gamma(1)$. Let $\{t_o = \gamma(s_o), t_1 = \gamma(s_1),\ldots,t_N = \gamma(s_N)\} \subseteq \Gamma$ ($0 = s_o < s_1 < \ldots < s_{N-1} < s_N = 1$) be a given set of points. We suppose a and b to be continuous on $\Gamma \setminus \{t_o,t_1,\ldots,t_N\}$.

In what follows we consider partitions $\Delta = \{\sigma_o, \sigma_1,\ldots,\sigma_n\}$ $(0 = \sigma_o < \sigma_1 \ldots < \sigma_n = 1)$ of the interval $[0,1]$, which contain $\{s_o,s_1,\ldots,s_N\}$, and we set $h_\Delta := \max \{\sigma_{i+1} - \sigma_i \mid i = 0,\ldots,n-1\}$. Now we define the space $PS_d(\Delta,\mathbb{C}^m)$ of all 'piecewise continuous' splines of degree d subordinate to the partition Δ : $PS_d(\Delta,\mathbb{C})$ consists of all $\varphi \in PC(\Gamma,\mathbb{C})$ such that $\varphi \circ \gamma$ is (d-1)times continuously differentiable on $[0,1] \setminus \{s_o,\ldots,s_N\}$ and the restriction of $\varphi \circ \gamma$ to $[\sigma_i,\sigma_{i+1}]$ is a polynomial of degree not greater than d – i.e. the functions $\varphi \in PS_d(\Delta,\mathbb{C})$ are splines with maximal smoothness at $\sigma_i \in \Delta \setminus \{s_o,\ldots,s_N\}$ and no conditions on smoothness at $s_i (i = o,\ldots,N)$. Obviously, $PS_o(\Delta,\mathbb{C})$ is the space of piecewise constant functions subordinate to the partition Δ . By $PS_d(\Delta,\mathbb{C}^m)$ we denote the space of vector-valued functions $f=(f_1 \dots, f_m)$ $\in PC(\Gamma,\mathbb{C}^m)$ with components $f_j \in PS_d(\Delta,\mathbb{C})$.

The spline Galerkin method for approximate solving the equation $Ax=y$ now reads as: Find $x_\Delta \in PS_d(\Delta,\mathbb{C}^m)$ such that

$$(Ax_\Delta, v) = (y,v), \ \forall v \in PS_d(\Delta,\mathbb{C}^m) \tag{3.1}$$

The linear algebraic system of Galerkin equations (3.1) is equivalent to the projection equation

$$P_\Delta AP_\Delta x_\Delta = P_\Delta y, \tag{3.2}$$

where P_Δ stands for the orthogonal projection of $L^2(\Gamma,\mathbb{C}^m)$ onto the subspace $PS_d(\Delta,\mathbb{C}^m)$.

Let $\{\Delta_k\}_{k\in\mathbb{N}}$ be a sequence of partitions of $[0,1]$. Such a sequence will be called <u>admissible</u> if $\{s_0,s_1,\dots,s_N\} \subseteq \Delta_k (k\in\mathbb{N})$ and $\hbar_{\Delta_k} \longrightarrow 0 (k\to\infty)$. We shall say that the Galerkin method with respect to the sequence of partitions $\{\Delta_k\}$ is <u>convergent</u> for the operator A if (3.1) is uniquely solvable for $\Delta=\Delta_k$ and sufficiently large k and if x_{Δ_k} converges to $x=A^{-1}y$ in $L^2(\Gamma,\mathbb{C}^m)$ for any $y\in L^2(\Gamma,\mathbb{C}^m)$.

<u>Theorem 2. Let $\{\Delta_k\}$ be an arbitrary admissible sequence of partitions and let $A=aP_\Gamma+bQ_\Gamma \in \mathcal{L}(L^2(\Gamma,\mathbb{C}^m))$ be invertible and strongly elliptic. Then the Galerkin method (3.1) with respect to the sequence $\{\Delta_k\}$ is convergent.</u>

The proof rests on the well-known theory of projection methods of type (3.2) (see e.g. [5]) and on the following lemma.

<u>Lemma 2.</u> Suppose $\{\Delta_k\}$ to be an admissible sequence of partitions and $f\in PC(\Gamma,\mathbb{C}^{m\times m})$ to be continuous on $\Gamma\setminus\{t_0,t_1,\dots,t_N\}$. Then

$$\|(I-P_{\Delta_k})fP_{\Delta_k}\| \longrightarrow 0, \quad \|P_{\Delta_k}f(I-P_{\Delta_k})\| \longrightarrow 0$$

as $k\to\infty$.

Lemma 2 was independently stated in [6], [7] and in [1] in the case of smooth functions f. In these papers, however, the authors considered arbitrary Sobolev norms instead of L^2-norms and therefore the partitions were supposed to be regular. The proof of Lemma 2 is based on an idea used in proving Lemma 4.1 [10] and on well-known properties of spline basis functions (see [2], [3]).

In the case m = 1, Theorem 2 is convertible:
Theorem 3. Let m = 1 and $a, b \in PC(\Gamma, \mathbb{C})$. If the Galerkin method (3.1) is convergent for the operator $A = aP_\Gamma + bQ_\Gamma \in \mathcal{L}(L^2(\Gamma, \mathbb{C}))$ with respect to an admissible sequence of (equi-distant) partitions Δ_k, then A is strongly elliptic.

The proof of Theorem 3 rests on the method of associated operators for spline approximation developed in the authors' paper (see [9]).

4. A spline collocation method. We suppose Γ to be closed and select the sequence of equidistant partitions $\Delta_k = \{\sigma_0, \sigma_1, \dots, \sigma_k\}$ with $\sigma_j = j/k$ $(j = 0, 1, \dots, k)$. In the method of piecewise linear collocation we determine a piecewise linear approximation $x_{\Delta_k} \in S_1(\Delta_k, \mathbb{C}^m)$ of the solution x of Ax = y from the collocation equations

$$(Ax_{\Delta_k})(\gamma(\tfrac{j}{k})) = y(\gamma(\tfrac{j}{k})), \; j = 0, \dots, k-1 \tag{4.1}$$

The linear algebraic system (4.1) is equivalent to the projection equation

$$K_{\Delta_k} A x_{\Delta_k} = K_{\Delta_k} y .$$

Here K_{Δ_k} stand for the projection of $L^2(\Gamma, \mathbb{C}^m)$ onto $S_1(\Delta_k, \mathbb{C}^m)$ defined by piecewise linear interpolation with the knots $\gamma(\frac{j}{k})$.

It is well known that the system (4.1) is uniquely solvable for sufficiently large k and $x_{\Delta_k} \longrightarrow x$ in $L^2(\Gamma, \mathbb{C}^m)$ for $k \to \infty$ and for any $y \in C(\Gamma, \mathbb{C}^m)$, if the collocation method is stable, i.e. if

$$\| K_{\Delta_k} A \varphi \| \geq c \|\varphi\| \; , \; \forall \varphi \in S_1(\Delta_k, \mathbb{C}^m),$$

with c(const) > 0.

One of the main results for the collocation method can be formulated as follows.

Theorem 4. (a) [9]: Let $a,b \in PC(\Gamma, \mathbb{C}^{m\times m})$. If the invertible operator $A = aP_\Gamma + bQ_\Gamma \in \mathcal{L}(L^2(\Gamma, \mathbb{C}^m))$ is strongly elliptic, then the collocation method (4.1) is stable.

(b) [8]: Let $m = 1$ and $a,b \in PC(\Gamma, \mathbb{C})$. Then the collocation method (4.1) is stable if and only if the operator $A = aP_\Gamma + bQ_\Gamma \in \mathcal{L}(L^2(\Gamma, \mathbb{C}))$ is strongly elliptic.

The proofs rest on certain localization techniques and on the method of associated operators for spline approximation (see [8], [7]).

Corollary. Let $m = 1$ or let a and b be continuous. Then the following conditions are equivalent for the invertible operator (1.1):

(i) the strong ellipticity condition
(ii) condition (B)
(iii) the Galerkin method (3.1) (with respect to any admissible sequence of partitions) converges for the operator (1.1)
(iv) the collocation method (4.1) is stable.

This is a consequence of the afore mentioned results, of Theorem 2 [9] and of Theorem 4.1 [8].

In [8] necessary and sufficient conditions are given for the stability of the piecewise linear collocation (4.1) in the case $m \geq 1$. Recently Schmidt [12], [13] obtained that for $m = 1$ the strong ellipticity condition is also necessary and sufficient for the stability of the collocation by means of arbitrary odd degree polynomial splines.

It seems to us to be an open, however interesting problem, whether the equivalence of conditions (i) to (iv) holds in the general case $a,b \in PC(\Gamma, \mathbb{C}^{m\times m})$ with $m > 1$.

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