КОНСТРУКТИВНАЯ ТЕОРИЯ ФУНКЦИЙ'84
София, 1984, с. 718—725

CONSTRUCTIVE THEORY OF FUNCTIONS'84
Sofia, 1984, p.718-725

UNIFORM APPROXIMATION ON THE WHOLE REAL LINE

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1. Introduction and statement of the result. Entire functions of exponential type are amongst the most important interpolants of a given function on the whole real line. As regards their approximating property it was shown by Bernstein [1] that a function $f : \mathbb{R} \to \mathbb{R}$ can be approximated arbitrarily closely and uniformly on the whole real line by entire functions of exponential type if and only if f is uniformly continuous and bounded.

In [4] the following interpolation operator was considered

$$R_\tau(f;\beta,z) := \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\tau}\right) A_n\left(\frac{\tau}{\pi}z\right) + \left(\frac{\pi}{\tau}\right)^2 \beta_{\tau n} B_n\left(\frac{\tau}{\pi}z\right) \quad ,$$

where $(\beta_{\tau n})_{n\in\mathbb{Z}}$ is any bounded sequence of complex numbers depending on a parameter $\tau > 0$ and A_n, B_n are the fundamental functions of (o,2)-interpolation satisfying the following conditions

(i) A_n and B_n are entire functions of exponential type 2π,

(ii) $A_n(k) = B_n''(k) = \delta_{nk}$ (Kronecker symbol),

$A_n''(k) = B_n(k) = 0$ for all $n,k \in \mathbb{Z}$,

(iii) $A_n'(0) = B_n'(0) = A_n'''(0) = B_n'''(0) = 0$ for all $n \in \mathbb{Z}$.

The properties (i)-(iii) uniquely determine the functions A_n and B_n; they turn out to be bounded on the real line by a constant not depending on n. The interpolant $R_\tau(f;\beta,z)$ exists whenever f is bounded and is in fact an entire function of exponential type 2τ. Concerning the convergence for $\tau \to \infty$ the following theorem was proved.

THEOREM A [4, Theorem 3]. *If* $f : \mathbb{R} \to \mathbb{R}$ *is a continuous and bounded function satisfying*

(1) $$f(x+h) - 2f(x) + f(x-h) = o(h)$$

uniformly in x *as* $h \to o$ *and*

$$\sup_n |\beta_{\tau n}| = o(\tau) \quad \text{as} \quad \tau \to \infty ,$$

then

(2) $$\lim_{\tau\to\infty} R_\tau(f;\beta,x) = f(x)$$

uniformly in x *on every compact subset of the real line.*

It may be pointed out that (2) does not hold in general even for a continuous function with compact support (see [3]). The purpose of this note is to present sufficient conditions ensuring uniform convergence of $R_\tau(f;\beta,x)$ to $f(x)$ on the whole real line. We shall see in particular that if f is continuously differentiable and for some $\alpha > 1$

(3) $$(1+|x|^\alpha)|f^{(j)}(x)| = O(1) \quad \text{for} \quad j = o,1$$

as $x \to \pm\infty$, then

(4) $$\lim_{\tau\to\infty} \sup_{x\in\mathbb{R}} |R_\tau(f;o,x) - f(x)| = o .$$

In fact, the following theorem holds.

THEOREM 1. *Let* $\alpha > 1$ *and* $\rho(x) := 1+|x|^\alpha$. *If*

(i) $\rho(x)|f(x)| = O(1)$ *as* $x \to \pm\infty$,

(ii) $\rho(x)(f(x+2h) - 2f(x+h) + f(x)) = o(h)$

uniformly in x *as* $h \to o$, *and*

(iii) $\sup_{n\in\mathbb{Z}} \rho(\frac{n\pi}{\tau})|\beta_{\tau n}| = o(\tau)$ *as* $\tau \to \infty$,

then

$$\lim_{\tau\to\infty} R_\tau(f;\beta,x) = f(x)$$

uniformly in x *on the whole real line.*

2. Lemmas. We will need the following auxiliary results.

LEMMA 1. *Let* f *be a function defined on* $\mathbb{R}$ *and let*

$$\omega_2(f;\alpha,t) := \sup_{x\in\mathbb{R}, |h|\le t} (1+|x|^\alpha)|f(x+2h) - 2f(x+h) + f(x)| ,$$

where $\alpha > 1$. *Then there exists a constant* C *depending only on* α *such that for all* $\eta > o$ *and* $t \in [o,1]$

$$\omega_2(f;\alpha,\eta t) \le C\cdot(1+\eta)^{2+\alpha}\omega_2(f;\alpha,t) .$$

Proof. We use the same ideas as in the case of $\omega_2(f;t)$ (see Lorentz [5, p. 47-48]). In the first step η is considered to be an integer n , say. Then

$$\Delta^2_{nt} f(x) = \sum_{k_1=o}^{n-1} \sum_{k_2=o}^{n-1} \Delta^2_t f(x+k_1t+k_2t)$$

and the inequality

$$|x|^\alpha \le 2^{\alpha-1}(|x+\xi|^\alpha + |\xi|^\alpha) ,$$

which is valid for every ξ , gives us the estimate

$$(1+|x|^\alpha)\Delta^2_{nt} f(x) \le$$

$$\sum_{k_1=o}^{n-1} \sum_{k_2=o}^{n-1} (1+2^{\alpha-1}(|x+k_1t+k_2t|^\alpha+|k_1t+k_2t|^\alpha))|\Delta^2_t f(x+k_1t+k_2t)|$$

$$\le 2^{\alpha-1} n^2 \omega_2(f;\alpha,t) + 2^{2\alpha-1} n^{2+\alpha} \omega_2(f;t)$$

$$\le 2^{2\alpha} n^{2+\alpha} \omega_2(f;\alpha,t) .$$

The remainder of the argument can be completed as for $\omega_2(f;t)$.

LEMMA 2. *Let* $f : \mathbb{R}\to\mathbb{R}$ *be a bounded, continuous function and let* $\alpha \in (1,2)$. *Then, for every* $\tau \ge 1$, *there exists an entire function* $Q_\tau(f;.)$ *of exponential type* τ *such that*

$$\sup_{x\in\mathbb{R}} (1+|x|^\alpha)|f(x)-Q_\tau(f;x)| \le \gamma\cdot\omega_2(f;\alpha,\tfrac{1}{\tau}) ,$$

where γ *is a constant depending only on* α .

Proof. Let

$$G_\tau(z) := (\frac{\sin \tau z/6}{z})^6 ,$$

$$\Omega_\tau := \int_{-\infty}^{\infty} G_\tau(x)dx ,$$

and define

$$Q_\tau(f;z) := \Omega_\tau^{-1} \int_{-\infty}^{\infty} f(t)(2G_\tau(z-t) - \frac{1}{2} G_\tau(\frac{z-t}{2}))dt .$$

Then $Q_\tau(f;.)$ is an entire function of exponential type τ (see [6, pp. 257-259]) which may also be written as

$$Q_\tau(f;x) = \Omega_\tau^{-1} \int_{-\infty}^{\infty} G_\tau(t)(2f(x+t) - f(x+2t))dt$$

provided $x \in \mathbb{R}$. Hence

$$(1+|x|^\alpha)|f(x)-Q_\tau(f;x)| \le \Omega_\tau^{-1} \int_{-\infty}^{\infty} G_\tau(t)\omega_2(f;\alpha,t)dt$$

$$\le C\, \Omega_\tau^{-1}\, \omega_2(f;\alpha,\tfrac{1}{\tau}) \int_{-\infty}^{\infty} G_\tau(t)(1+\tau t)^{2+\alpha}dt ,$$

where Lemma 1 is used in the last step. Now we note that

$$\int_{-1/\tau}^{1/\tau} G_\tau(t)(1+\tau t)^{2+\alpha}dt \le 2^{2+\alpha} \int_{-1/\tau}^{1/\tau} G_\tau(t)dt \le 2^{2+\alpha}\,\Omega_\tau$$

and

$$\left(\int_{-\infty}^{-1/\tau} + \int_{1/\tau}^{\infty} \right) G_\tau(t)(1+\tau t)^{2+\alpha}dt \le (2\tau)^{2+\alpha} \int_{-\infty}^{\infty} G_\tau(t)t^{2+\alpha}dt .$$

Since $\Omega_\tau = O(\tau^5)$ as $\tau \to \infty$ the desired result becomes obvious.

LEMMA 3. <u>Let</u> g <u>be an entire function of exponential type</u> $\tau > 1$ <u>such that</u>

$$(1+|x|^\alpha)|g(x)| \le M$$

<u>on</u> $\mathbb{R}$, <u>where</u> $\alpha \in (1,2)$. <u>Then there exists a constant</u> K <u>depending only on</u> α <u>such that</u>

(5) $$(1+|x|^\alpha)|g'(x)| \le \tau K\cdot M .$$

<u>Proof.</u> The function

$$h : z \mapsto (z+i)^\alpha g(z)$$

is holomorphic and of exponential type τ in the closed upper half plane and bounded by M on the real line. Hence [2, p. 82, (6.2.4)]

$$|h(x+iy)| \le Me^{\tau y} \quad \text{for all} \quad y \ge o$$

which implies that

$$|g(x+iy)| \le \frac{M}{1+|x|^{\alpha}} 2^{1-\alpha/2} e^{\tau|y|} .$$

Clearly, the same estimate holds for negative y . Now, using Cauchy's integral formula for $g'(x)$ and integrating along the circle of radius $1/\tau$ centered at x we arrive at (5) with $K \le e(2^{\alpha/2} + 2^{1-\alpha/2})$.

LEMMA 4. _Let_ $\alpha \in (1,2)$ _and_ $\rho(x) := 1+|x|^{\alpha}$. _If_ f _is a function defined on_ $\mathbb{R}$ _and satisfying_

(6) $$\rho(x)|f(x)| = O(1) \quad \text{as} \quad x \to \pm\infty$$

and

$$\omega_2(f;\alpha,h) = o(h) \quad \text{as} \quad h \to o ,$$

then there exists a sequence of entire functions $U_{\tau}(f;.)$ _of exponential type_ τ _such that for large_ τ

(7) $$\rho(x)|f(x) - U_{\tau}(f;x)| = o(1/\tau) ,$$

(8) $$\rho(x)|U'_{\tau}(f;x)| = o(\log \tau) ,$$

(9) $$\rho(x)|U''_{\tau}(f;x)| = o(\tau) ,$$

(10) $$\rho(x)|U'''_{\tau}(f;x)| = o(\tau^2)$$

uniformly in x .

Proof. Let $Q_{\tau}(f;.)$ be as in Lemma 2. For $\tau \in [2^k,2^{k+1})$, where $k \ge o$ is an integer, set

$$U_{\tau}(f;x) := Q_{2^k}(f;x) .$$

Then it is clear that (7) holds. In order to prove (8)-(10) we note that (for $\tau > 1$)

(11) $$U_{\tau}(f;x) = \sum_{j=1}^{k} (Q_{2^j}(f;x) - Q_{2^{j-1}}(f;x)) + Q_1(f;x) .$$

From Lemma 2 we have

$$\rho(x)|Q_{2^j}(f;x) - Q_{2^{j-1}}(f;x)| \le$$

$$\rho(x)(|f(x)-Q_{2^j}(f;x)| + |f(x)-Q_{2^{j-1}}(f;x)|) = o(2^{-j})$$

uniformly in x. Using (6) we find

$$\rho(x)|Q_1(f;x)| \le \rho(x)(|f(x)-Q_1(f;x)| + |f(x)|) = O(1) .$$

By virtue of Lemma 3 we obtain

$$\rho(x)|Q'_{2^j}(f;x) - Q'_{2^{j-1}}(f;x)| = o(1) ,$$

$$\rho(x)|Q''_{2^j}(f;x) - Q''_{2^{j-1}}(f;x)| = o(2^j) ,$$

$$\rho(x)|Q'''_{2^j}(f;x) - Q'''_{2^{j-1}}(f;x)| = o(2^{2j}) ,$$

and

$$\rho(x)|Q_1^{(\nu)}(f;x)| = O(1) \qquad (\nu = 1,2,3)$$

uniformly in x as $j \to \infty$. Now the proof is readily completed with the help of the representation (11) of $U_\tau(f;x)$.

3. Proof of the theorem. For the function $U_\tau(f;.)$ of Lemma 4 we may write (see [4, Theorem 2])

$$U_\tau(f;z) = \sum_{n=-\infty}^{\infty} (U_\tau(f;\tfrac{n\pi}{\tau})A_n(\tfrac{\tau}{\pi}z) + (\tfrac{\pi}{\tau})^2 U''_\tau(f;\tfrac{n\pi}{\tau})B_n(\tfrac{\tau}{\pi}z))$$

$$+ c_1 \sin \tau z + c_2 \sin 2\tau z ,$$

where

$$c_1 = \tfrac{1}{3}(\tfrac{4}{\tau} U'_\tau(f;o) + \tfrac{1}{\tau^3} U'''_\tau(f;o)) ,$$

$$c_2 = -\tfrac{1}{6}(\tfrac{1}{\tau} U'_\tau(f;o) + \tfrac{1}{\tau^3} U'''_\tau(f;o)) .$$

Hence

$$f(x) - R_\tau(f;\beta,x) = f(x) - U_\tau(f;x) + U_\tau(f;x) - R_\tau(f;\beta,x)$$

$$= f(x) - U_\tau(f;x) + \sum_{n=-\infty}^{\infty} (U_\tau(f;\tfrac{n\pi}{\tau}) - f(\tfrac{n\pi}{\tau}))A_n(\tfrac{\tau}{\pi}x)$$

$$+ (\tfrac{\pi}{\tau})^2 \sum_{n=-\infty}^{\infty} (U''_\tau(f;\tfrac{n\pi}{\tau}) - \beta_{\tau n})B_n(\tfrac{\tau}{\pi}x)$$

$$+ c_1 \sin \tau z + c_2 \sin 2\tau z .$$

Next we may clearly assume without loss of generality that $\alpha \in (1,2)$.

Then from Lemma 4 we obtain

$$|U_\tau(f;\tfrac{n\pi}{\tau}) - f(\tfrac{n\pi}{\tau})| = \frac{1}{\rho(n\pi/\tau)}\, o(\tfrac{1}{\tau}) ,$$

$$|U_\tau''(f;\tfrac{n\pi}{\tau}) - \beta_{\tau n}| = \frac{1}{\rho(n\pi/\tau)}\, o(\tau) ,$$

and

$$c_j = o(1) \qquad (j = 1,2) .$$

Now, recalling the boundedness of the fundamental functions A_n and B_n on the real line the desired result follows from the fact that

$$\frac{1}{\tau}\sum_{n=-\infty}^{\infty} \frac{1}{\rho(n\pi/\tau)} = O(1)$$

as $\tau \to \infty$.

4. Two special cases. Now as a corollary we first deduce that (3) implies (4).

By the mean value theorem

$$f(x+2h) - 2f(x+h) + f(x) = hf'(x+h+\vartheta_1 h) - hf'(x+\vartheta_2 h) ,$$

where $\vartheta_1,\vartheta_2 \in (o,1)$. Hence in view of (3)

$$\Delta_2(f;\tfrac{1+\alpha}{2},x) := (1+|x|^{\frac{1+\alpha}{2}})|f(x+2h) - 2f(x+h) + f(x)|$$

$$\le h\, O(|x|^{\frac{1-\alpha}{2}}) \qquad \text{as} \quad x \to \pm\infty.$$

Thus, given $\varepsilon > o$ we can find a positive number x_o such that

(12) $$\Delta_2(f;\tfrac{1+\alpha}{2},x) \le \varepsilon h \qquad \text{for} \quad x \notin [-x_o,x_o]$$

Further, f' being continuously differentiable there exists an $h_o > o$ with the property that for all $h \in (o,h_o]$ and $x \in [-x_o-h_o,x_o+h_o]$

$$\left|\frac{f(x+h)-f(x)}{h} - f'(x)\right| < \frac{\varepsilon}{3}(1+|x_o|^{\frac{1+\alpha}{2}})^{-1}$$

and

$$|f'(x) - f'(x+h)| < \frac{\varepsilon}{3}(1+|x_o|^{\frac{1+\alpha}{2}})^{-1} .$$

Hence

(13) $$\Delta_2(f;\tfrac{1+\alpha}{2},x) \le \varepsilon h \qquad \text{for} \quad x \in [-x_o,x_o] , \quad h \in (o,h_o] .$$

Now (12) and (13) imply that condition (ii) of Theorem 1 is satisfied with α replaced by $\frac{1+\alpha}{2}$.

Secondly we remark that if the function f of Theorem A is of compact support, then the conditions of Theorem 1 are satisfied, provided the sequence $(\beta_{\tau n})_{n\in\mathbb{Z}}$ is chosen appropriately. Hence in this case (2) holds uniformly on the whole real line.

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