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AN ANGLE IN $L^2(\mathbb{C})$ DETERMINED BY TWO PLANE DOMAINS

I.P. Ramadanov and M.L. Skwarczyński

The Hilbert space methods in the theory of several complex variables were introduced in the works of Stefan Bergman [2,3]. In these methods an important role is played by the Bergman function $K_D(z,t)$ - the reproducing kernel in the Hilbert space $L^2H(D)$ of all square integrable and holomorphic functions in a domain $D \subset \mathbb{C}^n$. This function is a source of numerous applications. Naturally one would like to have some means to compute the function $K_D(z,t)$ for a general domain $D \subset \mathbb{C}^n$. Of course, in a general case, we can not expect a closed analytic formula, like the one for the ball. In the present paper we shall recall a procedure [6] which allows us to construct $K_D(z,t)$, $z \in D$ as an element in $L^2(D)$. Actually, using the functions K_{D_i}, $i = 1,2,\ldots,m$ one can form a sequence, which in $D = D_1 \cup D_2 \cup \ldots \cup D_m$ is L^2 - convergent to K_D. It follows that K_D can be constructed when D is a union of finitely many balls. A general domain $D \subset \mathbb{C}^n$ can be represented as a union of an increasing sequence D_k, $k = 1,2,\ldots$, where every D_k is a finite union of balls. Then K_D can be constructed from the functions K_{D_k} according to [5]. The above procedure follows as a simple corollary from a general theorem of I. Halperin [4] about projections in an abstract Hilbert space. Therefore it has wider scope and remains valid in other spaces with kernel functions which are contained in $L^2(D)$.

For $\mathbb{C} = D_1 \cup D_2$, where each D_i, $i = 1,2$ is a halfplane the process of constructing $K_{\mathbb{C}} = 0$ can be illustrated by immediate calculations. This leads to some new orthogonality relations [6]. Finally, we shall gain some insight into this procedure by studying the angle γ between corresponding subspaces in $L^2(D)$ in the case when $D = \mathbb{C}$.

1. Alternating projections. The following result was proved in [7]. (A standard reference for the Bergman function is [1]).

THEOREM 1. Let $D \subset \mathbb{C}^n$ be the union of the domains D_i, $i = 1, \ldots, m$ (with known Bergman functions). In the Hilbert space $H = L^2(D)$ we consider for each i the closed linear subspace F_i of all elements which are holomorphic in D_i. Let $P_i : H \to F_i$ be the corresponding orthogonal projection. Let $t \in D$ ($t \in D_1$ with no loss of generality). Consider $f \in H$ defined by

$$f(z) = \begin{cases} K_{D_1}(z,t) & , z \in D_1 \\ 0 & , z \in D \setminus D_1 \end{cases}$$

Then the sequence of alternating projections

$$f_1 = P_1 f, \quad f_2 = P_2 f_1, \ldots, f_m = P_m f_{m-1}, f_{m+1} = P_1 f_m, \ldots$$

converges in H to $K_D(\cdot,t)$.

The proof can be easily derived from the following result due to I. Halperin [4]:

Let H be a Hilbert space and P_i the projection of H onto a subspace $F_i \subset H$, $i = 1,2, \ldots, m$. Let $F = \bigcap F_i$ and denote by P the projection of H onto F. Then

$$\lim (P_m P_{m-1} \cdot \ldots P_1)^n f = Pf$$

for every $f \in H$.

In this paper we shall consider the simplest case when $m = 2$, $n = 1$, $D = \mathbb{C}$. Then $F_1 \cap F_2 = \{0\}$. Assuming that $\operatorname{int}(D_1 \setminus D_2) \neq \emptyset$ and $\operatorname{int}(D_2 \setminus D_1) \neq \emptyset$ we shall study the angle $\gamma \in [0, \pi/2)$ between F_1 and F_2. By definition

$$\text{(1)} \qquad \cos\gamma = \sup\left\{ \frac{|\langle f_1, f_2 \rangle|}{\|f_1\| \cdot \|f_2\|},\ f_i \in F_i \setminus \{0\},\ i = 1,2 \right\}.$$

For a future use, let us mention the following orthogonality condition [7]: for every $f_1 \in F_1$ and every $f_2 \in F_2$

$$\text{(2)} \qquad \int_{D_1 \cap D_2} f_1 \bar{f}_2 = 0 .$$

2. Reduction to an extremal problem. Since $\mathbb{C}$ is connected the open set $T = D_1 \cap D_2$ is nonvoid. Consider the following extremal problem in D_i: find

$$p_i = \sup\left\{ \frac{\|h\|_{D_i \setminus T}}{\|h\|_{D_i}} , \ h \in L^2H(D_i) \setminus \{0\} \right\} .$$

Note that $0 < p_i \leq 1$. We can now state:

THEOREM 2. If (2) holds, then

$$\cos \gamma = \max\{p_1, p_2\} . \tag{3}$$

The proof is obtained considering three cases of behaviour for the pairs (f_1, f_2) admissible in (1). The first case is when we assume additionally that $f_1 \equiv 0$ on D_1. Using some direct computations and the Schwarz inequality, we get that the supremum in (1) is equal to p_2. If the above condition is replaced by $f_2 \equiv 0$ on D_2, the reasoning is similar and the obtained value is p_1. The second case concerns the assumption that $f_1 \equiv 0$ on $D_2 \setminus T$. In this case, via the orthogonality condition (2), it is easy to obtain that the supremum is p_1. A similar reasoning shows that the supremum (1) under the condition $f_2 \equiv 0$ on $D_1 \setminus T$ is p_2. The last step is that we assume neither the first case nor the second hold. Here the computations are longer and in fact, they show that the supremum is obtained already in the above cases.

3. Domains bounded by concentric circles. Let us consider the case when D_1 is the interior of a disc, and D_2 is the exterior of a smaller concentric disc. Applying (if necessary) an automorphism of $\mathbb{C}$ we may assume that for some $r \in (0,1)$

$$D_1 = \{ z \in \mathbb{C} : |z| < 1 \} \qquad , \qquad D_2 = \{ z \in \mathbb{C} : |z| > r \} .$$

To determine p_1, note that for $n = 0, 1, \ldots$, the monomials z^n of different degrees are orthogonal over every disc centered at the origin, and for every $f(z) = \sum a_n z^n$ in $L^2H(D_1) \setminus \{0\}$,

$$\frac{\|f\|^2_{D_1 \setminus T}}{\|f\|^2_{D_1}} \leq r^2$$

This implies $p_1 = r$.

The computation of p_2 is similar. It is well known, that the monomials z^{-n}, $n = 2,3,\ldots$ are orthogonal and linearly dense in $L^2H(D_2)$. Moreover,

$$\frac{\|z^{-n}\|_{D_2 \setminus T}}{\|z^{-n}\|_{D_2}} = r^{\frac{2n-2}{2}} \leq r$$

with equality for n = 2. Using the Laurent development we obtain similarly as before that $p_2 = r$.

The orthogonality condition (2) in our case is obviously satisfied. Therefore by THEOREM 2

$$\cos \gamma(r) = r \quad .$$

In particular,

$$\lim_{r \to 1} \gamma(r) = 0 .$$

4. Domains bounded by parallel lines. We consider now the halfplanes D_1 and D_2 which intersect along a strip T of width s. With no loss of generality, let us assume that

$$D_1 = \{ z \in \mathbb{C} : \operatorname{Re} z > 0 \} \quad , D_2 = \{ z \in \mathbb{C} : \operatorname{Re} z < s \} \quad .$$

It was proved in [6] that the orthogonality condition (2) holds in this case. Also $p_1 = p_2$ for reason of symmetry. It remains to compute p_1. This problem is clearly invariant under biholomorphic mapping of D_1 (one has to apply the canonical isometry , see [8] p. 21, associated with this mapping). A linear fractional transformation maps D_1 onto the unit disc Δ in such a way that $D_1 \setminus T$ is mapped onto the smaller disc V, internally tangent to Δ at 1. Consider now a still smaller disc U, contained in V, with fixed radius and variable center on the real axis. For every $f \in L^2H(\Delta) \setminus \{0\}$ we have an obvious inequality

$$(4) \qquad \frac{\|f\|_U^2}{\|f\|_\Delta^2} \leq \frac{\|f\|_V^2}{\|f\|_\Delta^2} \leq p_1^2 \quad .$$

Consider now an automorphism of the unit disc Δ which maps U onto the disc U_r with center at 0 and radius r. Note that r depends on the position of U, and approaches 1 when U moves toward the position of tangency with Δ. The result in section 3 yields

$$r^2 = \sup_{h\in L^2H(\Delta)} \frac{\| h\|^2_{U_r}}{\| h\|^2_{\Delta}} = \sup_{f\in L^2H(\Delta)} \frac{\| f\|^2_{U}}{\| f\|^2_{\Delta}}$$

In view of (4), the right side is not greater than p_1^2. Hence $r \leq p_1$. We can now move U toward the position of tangency, and conclude that $p_1 = 1$. By THEOREM 2, $\cos \gamma = 1$. Therefore, for all s

$$\gamma = 0 .$$

This shows that domains bounded by parallel lines present rather unfavorable case for THEOREM 1. Nevertheless (as it was shown in [6]), the convergence discussed in this theorem is still satisfactory.

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Institute of Mathematics
Bulgarian Academy of Sciences
P.O.Box 373 1090 Sofia Bulgaria

ul. Smoleńskiego 27 a
Warszawa 01689
Poland