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CONSTRUCTING (BETTER) TRIGONOMETRICAL KERNELS BY REMOVING ZEROES

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1. Definitions. The background of the following considerations is the approximation of 2π-periodic functions (e.g. $f \in C_{2\pi}$) by means of singular convolution integrals

(1) $$I_n(p;f;x) := (f*p_n)(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)p_n(t)dt \qquad (\mathbb{N} \ni n \to \infty)$$

where the kernel p_n is a normalized, even, trigonometrical polynomial of degree n

(2) $$\text{(i)}\ p_n(x) := \frac{1}{2} + \sum_{k=1}^{n} \rho_{k,n}(p)\cos kx\,, \qquad \text{(ii)}\ \int_0^{\pi} p_n(t)dt = \frac{\pi}{2} \qquad (n \in \mathbb{N}).$$

In case $p_n(x) \geq 0$ the notation is $p_n \in N\Pi_n^+$; in general, $p_n \in N\Pi_n^{\pm}$.

The linear operator (1) establishes a strong approximation process

(3) $$\lim_{n\to\infty} \| I_n(p;f;\circ) - f(\circ)\| = 0 \qquad (\|f(\circ)\| := \sup_{x\in\mathbb{R}} |f(x)|)$$

provided the kernel (2) is an approximate identity (≡apid), i.e.,

(4) $$\text{(i)}\ \int_0^{\pi} |p_n(t)|dt \leq M \ (n \in \mathbb{N}), \quad \text{(ii)}\ \lim_{n\to\infty} \int_{\delta\leq t\leq\pi} |p_n(t)|dt = 0 \quad (0<\delta<\pi)\ ;$$

condition (4,ii) on the kernel itself may be replaced by the condition

$$\lim_{n\to\infty} \rho_{k,n}(p) = 1 \qquad (k=1,2,3...)$$

on the convergence factors (Fourier coefficients) of (2), namely

$$\rho_{k,n}(p) := \frac{2}{\pi}\int_0^{\pi} p_n(t)\cos kt\,dt \qquad (1\leq k\leq n),$$

$$\rho_{0,n}(p) \equiv 1 \quad (n \in \mathbb{N}), \quad \rho_{k,n}(p) = 0 \quad (k > n).$$

The degree of approximation of (1) - or in (3), respectively - is controlled via the saturation limit

$$\lim_{n\to\infty} n^{\tau}\left(1 - \rho_{k,n}(p)\right) = \psi(k) \qquad (\tau > 0; k=1,2,3,\ldots)$$

[with $\psi(k)$ being a certain algebraic polynomial in k which characterizes the saturation class corresponding to (1) - a problem not of further interest in this connection], thus with $O(n^{-\tau})$, $n\to\infty$, giving the optimal (saturation) order of approximation of (1) induced by the apid (2). - For more details of all of these preliminaries see e.g. [6],[8].

2. Motivation. The starting point is the fact that an apid should be an approximating sequence for the δ-"function", thus having the peaking property at the origin $x=0 \pmod{2\pi}$, i.e.,

$$\max_{-\pi<x\leq\pi} p_n(x) = p_n(0) > 0 \qquad (n \in \mathbb{N})$$

(or, at least, in a close neighbourhood of $x=0$; cf. [1]). Concerning the graphical behaviour of an apid, the area under the corresponding graph should be concentrated near that peaking point, i.e., introducing the measure of condensation

$$C_n(\delta) := \int_{0<t<\delta} |p_n(t)|\,dt \qquad (0 < \delta < \pi),$$

condition (4,ii) may (also) be interchanged by the more intuitive - in comparison with (2,ii)-property

$$\lim_{n\to\infty} C_n(\delta) = \frac{\pi}{2} \qquad (0 < \delta < \pi).$$

Having in mind this conception the following procedure suggests itself (roughly speaking): Take a (suitable) trigonometrical polynomial, remove certain (one or more, also with respect to the multiplicity) zeroes near to the origin, and normalize the resulting polynomial! In order to show what happens, thus to give a striking reasoning tbe following parallel treatment is to be discussed in some detail.

The input is an outmost simple polynomial together with another one generated via an equally simple translation $\left(q_n(x) \equiv p_n(x \pm \frac{\pi}{n})\right)$, both belonging to $N\Pi_n^+$:

(5)

$$p_n(x) := \frac{1}{2} - \frac{1}{2}\cos nx = \sin^2 \frac{nx}{2} \qquad\qquad q_n(x) := \frac{1}{2} + \frac{1}{2}\cos nx = \cos^2 \frac{nx}{2}.$$

Nevertheless, both of these polynomials possess a well-known extremal property ([12,p. 83]): *For* $p_n \in N\Pi_n^+$ *it holds that* $|\rho_{n,n}(p)| \leq \frac{1}{2}$; *equality is attained* (just) *for the polynomials of* (5). - The graphs of (5) are self-evident; see Fig. 1/2. The zeroes nearest to the origin (each, of course, of multiplicity 2) are given by

$$x_o = 0 \qquad\qquad x_o = \overset{+}{(-)}\frac{\pi}{n}$$

(*one* double zero) | (*two* symmetric double zeroes).

Removing these zeroes, thus defining new polynomials (with correspondingly lowered polynomial degree)

$$P_{n-1}(x) := \frac{p_n(x)}{1-\cos x} \geq 0 \quad (n \geq 1) \qquad\qquad Q_{n-2}(x) := \frac{q_n(x)}{(\cos\frac{\pi}{n} - \cos x)^2} \geq 0 \quad (n \geq 2)$$

should lead to the effect that the corresponding graphs are lifted in a neighbourhood of the origin whereas they are pushed down to the axis outside; see Fig. 3/4. - Indeed, there is an explosion of growth in $x=0$ (starting from $p_n(0)=0$, $q_n(0)=1$), however in quite a different way, namely

(7)

$$P_{n-1}(0) = \frac{n^2}{2} \qquad\qquad Q_{n-2}(0) = \frac{1}{4\sin^4\frac{\pi}{2n}} \simeq \frac{4}{\pi^4} n^4, \quad (n \to \infty)$$

This should be compared with the extremal property ([12,p. 83])

(8) $$\max_{t_n \in N\Pi_n^+} t_n(0) = \frac{n+1}{2}.$$

$$Q_{n-2}(\pm\frac{\pi}{n}) = \frac{n^2}{4\sin^2\frac{2\pi}{n}} \simeq \frac{1}{4\pi^2} n^4,$$

$$Q_{n-2}(0) > Q_{n-2}(\pm\frac{\pi}{n}).$$

Now, in order to control the area under the graphs of (6), normalization according to (2,ii) is appropriate:

$$\overline{P}_n(x) :\overset{!}{=} \nu(n)\cdot P_n(x) \qquad\qquad \overline{Q}_n(x) :\overset{!}{=} \mu(n)\cdot Q_n(x)$$

simultaneously the polynomial degree is made to fit $N\Pi_n^+$). This leads to the normalization factors

$$(9) \qquad \nu(n) = \frac{1}{n+1} \qquad\Big|\qquad \mu(n) = \frac{1}{n+2}\sin^2\frac{\pi}{n+2} \qquad (n \geq 0)$$

and thus finally to

$$(10) \qquad \overline{P}_n(x) = \frac{1}{2(n+1)}\left(\frac{\sin(n+1)\frac{x}{2}}{\sin\frac{x}{2}}\right)^2 \equiv F_n(x) \qquad\Big|\qquad \overline{Q}_n(x) = \frac{\sin^2\frac{\pi}{n+2}}{n+2}\left(\frac{\cos(n+1)\frac{x}{2}}{\cos\frac{\pi}{n+2} - \cos x}\right)^2 \equiv K_n(x)$$

which are, in the usual notation, nothing but the famous kernels (apid) of *Fejér* and of *Fejér-Korovkin*, respectively, with well-known convergence factors $(1 \leq k \leq n)$

$$(11) \qquad \rho_{k,n}(F) = 1 - \frac{k}{n+1} \qquad\Big|\qquad \rho_{k,n}(K) = \left(1 - \frac{k}{n+2}\right)\cos\frac{k\pi}{n+2} + \frac{1}{n+2}\cot\frac{\pi}{n+2}\sin\frac{k\pi}{n+2}$$

and saturation limits (with corresponding optimal approximation orders)

$$(12) \qquad \lim_{n\to\infty} n\left(1 - \rho_{k,n}(F)\right) = k \qquad\Big|\qquad \lim_{n\to\infty} n^2\left(1 - \rho_{k,n}(K)\right) = \frac{\pi^2}{2}k^2.$$

And both of them are extremal positive apid, too: F_n is uniquely determined by (8) - whereas $K_n(0) = \frac{1}{n+2}\cot^2\frac{\pi}{2(n+2)} \simeq \frac{4}{\pi^2}n$, $n \to \infty$ - and K_n is uniquely determined by the maximal first Fourier coefficient according to ([12,p. 83],[6, p. 84])

$$(13) \qquad \max_{p_n \in N\Pi_n^+} \rho_{1,n}(p) = \cos\frac{\pi}{n+2} = \rho_{1,n}(K),$$

thus being the optimal positive apid of $N\Pi_n^+$ at all (saturation order $O(n^{-2})$, smallest constant $\pi^2/2$ on the right-hand side of (12) for all of $N\Pi_n^+$).

Finally, this may be compressed into the résumé that the procedure of removing (suitable) zeroes and normalizing works, in order to build up well-behaved (and even better) apid.

Since the initial polynomials (5), as simple as they are, simultaneously generate two of the most important classical kernels the question arises whether there are further such kernels which may be constructed by the foregoing method. In rounding off, there are at least to immediate examples.

Starting from $r_n(x) := \cos nx$, thus a non-normalizable, oscillating polynomial of degree n, with first (simple) zeroes $x_0 = \overset{+}{(-)}\frac{\pi}{2n}$ the procedure yields

$$(14)\qquad \bar{R}_{n-1}(x) = \frac{1}{2}\sin\frac{\pi}{2n}\,\frac{\cos nx}{\cos\frac{\pi}{2n} - \cos x}\,,\quad \bar{R}_{n-1}(0) = \frac{1}{2}\cot\frac{\pi}{4n} \simeq \frac{2}{\pi}n\,,\quad n\to\infty$$

(with normalization factor as the leading coefficient); however, (14) is just the *Rogosinski* kernel with polynomial representation ($n \geq 1$)

$$\bar{R}_{n-1}(x) = \frac{1}{2} + \sum_{k=1}^{n-1} \rho_{k,n-1}(R)\cos kx \in N\Pi^{\pm}_{n-1},\quad \rho_{k,n-1}(R) = \cos\frac{k\pi}{2n}$$

and saturation limit

$$\lim_{n\to\infty} n^2\big(1 - \rho_{k,n}(R)\big) = \frac{\pi^2}{8}k^2 \qquad (k=1,2,3\ldots),$$

thus an (oscillating) apid with optimal approximation order $O(n^{-2})$, $n\to\infty$.

Whereas the parallel treatment of $g(x) := \sin nx$ does not apply since g is an odd function, the even polynomial

$$d_n(x) := \frac{1}{2}\,[\cos(n-1)x - \cos nx\,] = \sin\frac{x}{2}\sin(2n-1)\frac{x}{2},\ \int_0^{\pi} d_n(t)dt = 0 \qquad (n\in\mathbb{N})$$

with (the only double one) zero $x_0 = 0$ directly delivers the *Dirichlet* kernel (not an apid)

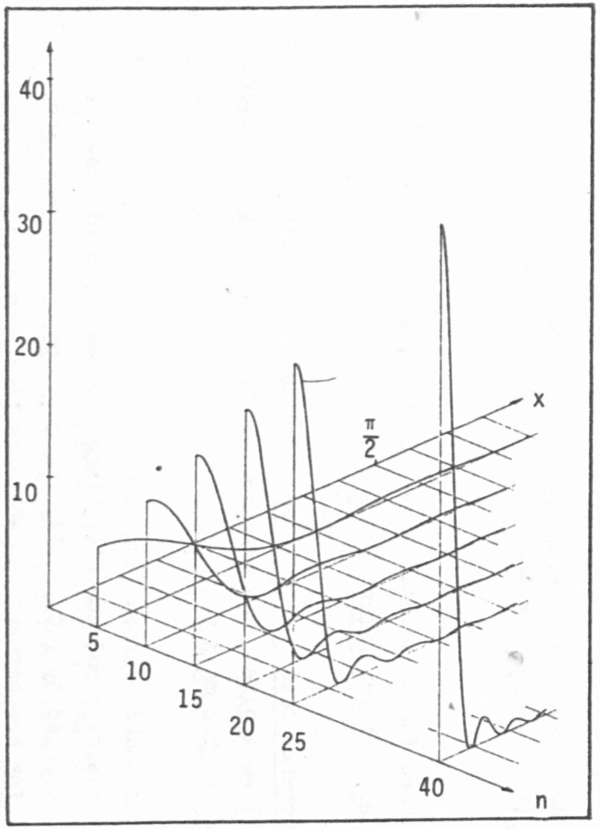
$$\bar{D}_n(x) = \frac{d_{n+1}(x)}{1-\cos x} = \frac{\sin(2n+1)\frac{x}{2}}{2\sin\frac{x}{2}} = \frac{1}{2} + \sum_{k=1}^{n}\cos kx \in N\Pi^{\pm}_n$$

(there is no need for additional normalization!) with $\bar{D}_n(0) = \frac{2n+1}{2}$, $\rho_{k,n}(D) \equiv 1$ ($1 \leq k \leq n$), i.e., without saturation limit.

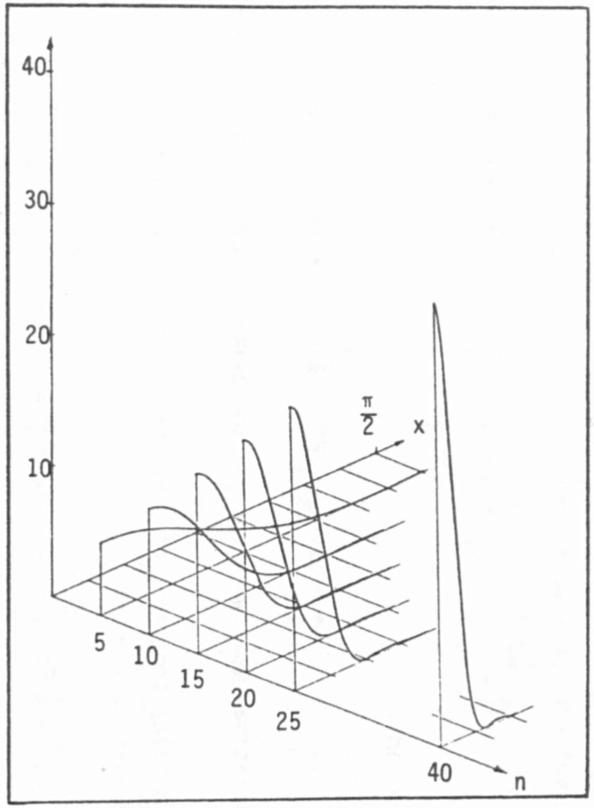
<u>3. The general case - an outline.</u> Due to the success of Sect. 2 the above procedure will be applied in a more general frame. To this end, starting e.g. from $t_n \in N\Pi_n^+$ there are two main cases.

<u>(I)</u> Removing *double* zeroes (and normalizing if possible, at all) obviously results anew in *positive* polynomials. Since the best possible positive kernel K_n has already been detected by (10) - (13) the question arises: what are the fine structure properties of the new kernels and what are they good for?

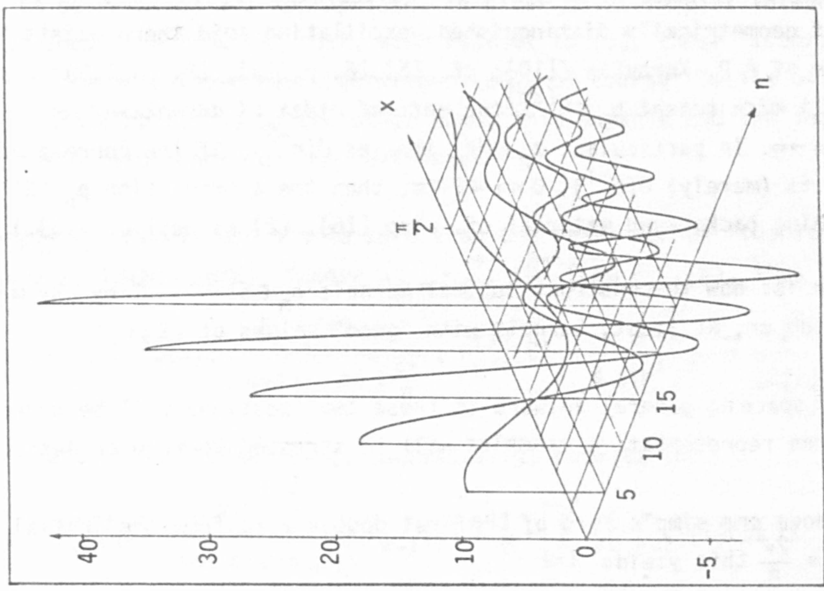
<u>(II)</u> Removing, say, 2m *simple* zeroes (symmetric; $m = 1,2,3,..$) from the given (at least) double zeroes of the initial $t_n(x) \geq 0$ leads to kernels of *finite 2m-oscillations*, i.e., of class $S^{(2m)}$.



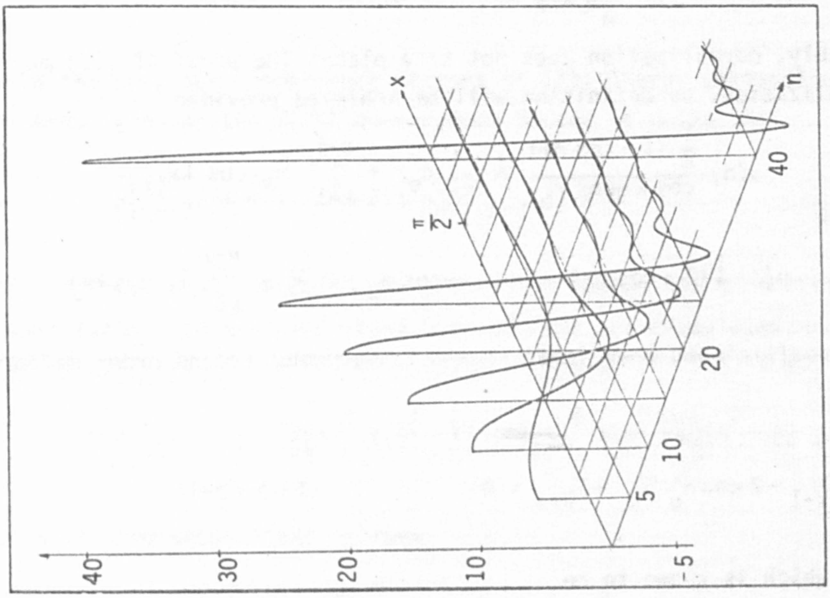
Kernel of Fejér



Kernel of Fejér-Korovkin



Kernel of Dirichlet



Kernel of Rogosinski

Stark

By definition: *Let* $t_n \in N\Pi_n^{\pm}$ *have a* *fixed* *number of* $2m$ *(symmetric) simple zeroes (or of odd multiplicity, resp.)* $(\overset{+}{-})x_j = x_j(n)$, $1 \le j \le m$, *for* $m = 1,2,3,\ldots$ *and* $m \ne m(n)$ *then* $t_n \in S^{(2m)}$. In particular: $N\Pi_n^{+} \equiv S^{(o)}$.

For this class of geometrically distinguished, oscillating apid there exists the *extension theorem* of P.P. Korovkin ([10]; cf. [5],[6, p. 93], [8, p. 120]: *For singular integrals* (1) *with kernel* $p_n \in S^{(2m)}$ *the optimal order of approximation is at most* $O(n^{-2-2m})$, $n \to \infty$. In particular: $p_n \in N\Pi_n^{+}$ implies $O(n^{-2})$; if the corresponding saturation order is (merely) $O(n^{-\tau})$, $0 < \tau \le 2+2m$, then the abbreviation $p_n \in S^{(2m,\tau)}$ is used. (Concerning background material cf. also [16], [2] as well as [13].)

The question then is: how to construct optimal kernels $p_n \in S^{(2m,2+2m)}$ by the method under consideration or, at least, kernels with "good" values of τ?

Due to economy of space no general answers to these two questions will be given; in place of that three representative examples will be stressed to more or less detail.

Example 1. Remove one simple zero of the first double zero from the initial $p_n(x)$ of (5). With $\sigma_o := \frac{2\pi}{n}$ this yields

$$(15) \qquad P^*_{n-1}(x) := \frac{\sin^2\frac{nx}{2}}{\cos x - \cos\sigma_o} = \sum_{k=1}^{n-1} \frac{\sin k\sigma_o}{\sin\sigma_o}\cos kx\,, \quad \int_0^{\pi} P^*_{n-1}(t)\,dt = 0,$$

i.e., unfortunately, normalization does not take place! The proof of (15) may be indicated: normalization, by definition, will be achieved provided

$$\nu(n)\cdot\frac{\frac{1}{2}(1-\cos nx)}{\cos x - \cos\sigma_o} \overset{!}{=} \frac{1}{2}\bar{\rho}_o + \sum_{k=1}^{n-1}\bar{\rho}_k \cos kx$$

$$\Longleftrightarrow$$

$$\nu(n)\,\frac{1}{2}(1-\cos nx) \overset{!}{=} (\cos x - \cos\sigma_o)\Big(\frac{1}{2}\bar{\rho}_o + \sum_{k=1}^{n-1}\bar{\rho}_k\cos kx\Big).$$

A comparison of coefficients then leads to the homogeneous, second order difference equation

$$(16) \qquad \bar{\rho}_{k-1} - 2\cos\sigma_o\cdot\bar{\rho}_k + \bar{\rho}_{k+1} = 0 \qquad (1 \le k \le n-1)$$

the solution of which is known to be

$$\bar{\rho}_k = \frac{\sin k\sigma_o}{\sin\sigma_o}\,, \quad \bar{\rho}_o = 0,\ \nu(n) \equiv 1 \quad (n \in \mathbb{N})$$

(see e.g. [9, p. 556], [14, p. 212]). Consequently, $P^*_{n-1} \notin N\Pi_{n+1}^{\pm}(\cap S^{(2)})$.

Nevertheless, on the one hand, this example points out the main difficulty: how to evaluate explicitly the normalization factor (if there is any)? In some cases difference equations of type (16) (or of higher order corresponding to the number of zeroes to be removed) are appropriate, in other cases complex integration applies, etc. (It should be remarked that, with respect to (9), these factors are readily at hand from the well-known representations (10), of course! Tables of integrals are of no use in this context.) On the other hand, this example and the ingredients of proof are of basic importance for the next one.

Example 2. Remove one simple zero from the Fejér kernel (10) itself, i.e., for any fixed j, $1 \leq j \leq E(n/2)$, remove $\alpha_{j,n} = \overset{+}{(-)} j\frac{2\pi}{n}$ from $F_{n-1}(x)$, thus investigate

$$(17) \qquad F_{n-2}^{[j]}(x) := \frac{F_{n-1}(x)}{\cos x - \cos j\frac{2\pi}{n}}\ ;\ F_{n-2}^{[j]}(0) = \frac{n}{4\sin^2 j\frac{\pi}{n}} \simeq \frac{n^3}{(2j\pi)^2}\ ,\ n\to\infty$$

(compare with (7)). Normalizing, successfully, results in

$$(18) \qquad \overline{F}_{n-2}^{[j]}(x) = \frac{2\sin^2 j\frac{\pi}{n}}{\cos x - \cos j\frac{2\pi}{n}} F_{n-1}(x) = \frac{1}{2} + \sum_{k=1}^{n-2} \rho_{k,n-2}^{[j]}(F)\cos kx \qquad (n \geq 2),$$

$$(19) \qquad \rho_{k,n-2}^{[j]}(F) = \rho_{k,n-1}(F) + \frac{1}{n}\,\frac{\sin kj\frac{2\pi}{n}}{\sin j\frac{2\pi}{n}} \qquad (1 \leq k \leq n-2),$$

involving the original convergence factors of (7). There, obviously, is a need for some remarks: (i) for the first convergence factor it holds that

$$(20) \qquad \rho_{1,n-2}^{[j]}(F) \equiv 1 \quad (n \geq 2);\ \rho_{n-1,n-2}^{[j]}(F) = 0$$

formally affirmes that, in fact, one polynomial degree has been lost; (ii) the additional factors on the right-hand side of (19) are calculated in accordance with (16) (for $j=1$); (iii) from the asymptotic expansion of (19), i.e.,

$$\rho_{k,n-2}^{[j]}(F) = 1 - \frac{(2j\pi)^2}{3!}\,\frac{k(k^2-1)}{n^3} + \frac{(2j\pi)^4}{5!}\,\frac{k(k^2-1)(k^2-3)}{n^5} + o(n^{-5}),\quad n\to\infty$$

the decisive saturation limit follows

$$(21) \qquad \lim_{n\to\infty} n^3\left(1-\rho_{k,n-2}^{[j]}(F)\right) = \frac{2}{3}\,j^2\pi^2 k(k^2-1) \qquad (k=2,3,4,\ldots)$$

(for $k=1$ see (20));here the particular saturation polynomial $k(k^2-1)$ of degree 3 appears; in any case, (21) ensures that (with $\tau=3$ instead of the optimal $\tau=4$)

$$F_{n-2}^{[j]} \in S^{(2,3)} ,$$

a *better* trigonometrical kernel. Moreover, (21) reveals by the leading constant on the right that the choice $j=1$ (the zero nearest to the peaking origin) is optimal; see also (17) for the growth behaviour in $x=0$. (iv) There is still quite another aspect: the closed representation of (18) may be rewritten via partial fractions as

$$(22) \qquad \bar{F}_{n-2}^{[j]}(x) = F_{n-1}(x) + \frac{1}{n} \frac{\sin^2 \frac{nx}{2}}{\cos x - \cos j\frac{2\pi}{n}} \qquad (n \geq 2)$$

$$= F_{n-1}(x) + \frac{1}{n} P_{n-1}^{*[j]}(x),$$

involving the polynomial $P_{n-1}^{*[j]}$ which corresponds to (15) for general j as given above $(\sigma_0(j) := j\frac{2\pi}{n})$. It should be noticed that both polynomials on the right-hand side are of degree (n-1). Furthermore, since $P_{n-1}^{*[j]}(0) = 0$ for all $n \geq 1$ it follows that

$$\bar{F}_{n-2}^{[j]}(0) \equiv F_{n-1}(0) = \frac{n}{2} \qquad (n \geq 2),$$

i.e., the peaking behaviour of the Fejêr kernel remaines invariant; the same remark holds for the normalization since by $\int_0^\pi P_{n-1}^{*[j]}(t)dt = 0$ for all admissible j, there is no contribution to the area under the graph of $\bar{F}_{n-2}(x)$. This, finally, reveals that kernels with increased order of approximation may also be constructed by *superposition*, and that of a known apid with a polynomial having properties like $P_{n-1}^{*[j]}$ (together with a suitable diminishing factor like $\frac{1}{n}$ in (22))! See Fig. 5 - 7. -

Summarizing: Example 2 gives a concrete partial answer to problem (II) for kernels of finite 2-oscillations; Example 1 exhibits that non-normalizable polynomials may occur by the underlying procedure, whereas, in turn, Example 2 shows what these polynomials are good for.

Example 3. Remove the first double zero (> 0) from the Fejêr kernel (or, equivalently, remove the first double zeroes $x_0 = 0$, $x_1 = \pm\frac{2\pi}{n}$ from $p_n(x)$ of (5)). After all, the procedure results in

$$(23) \qquad \bar{F}_{n-3}^{*}(x) := F_{n-1}(x) \frac{2\sin^2\frac{\pi}{n}\,\sin^2\frac{2\pi}{n}}{(3-2\sin^2\frac{\pi}{n})(\cos x - \cos\frac{2\pi}{n})^2} \in N\Pi_{n-3}^{+} \qquad (n \geq 3);$$

the peaking behaviour is now characterized by

Fig.1: $p_n(x)$, n=10

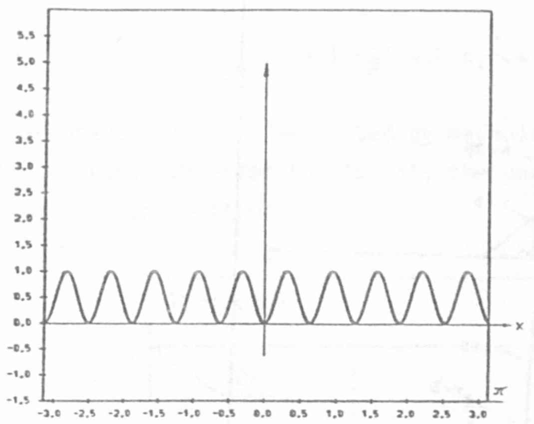


Fig.2: $q_n(x)$, n=10

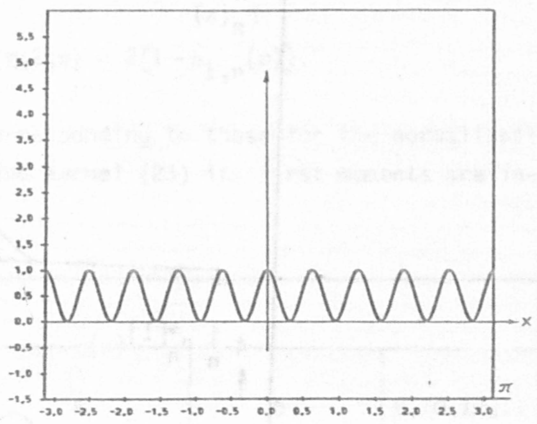


Fig.3: $p_n(x) \rightarrow F_{n-1}(x)$, n=4

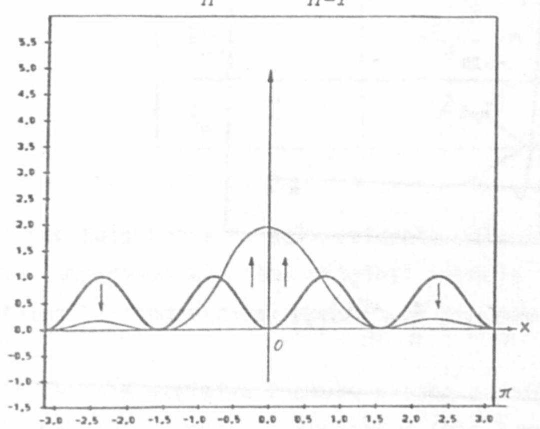


Fig.4: $q_n(x) \rightarrow K_{n-2}(x)$, n=4

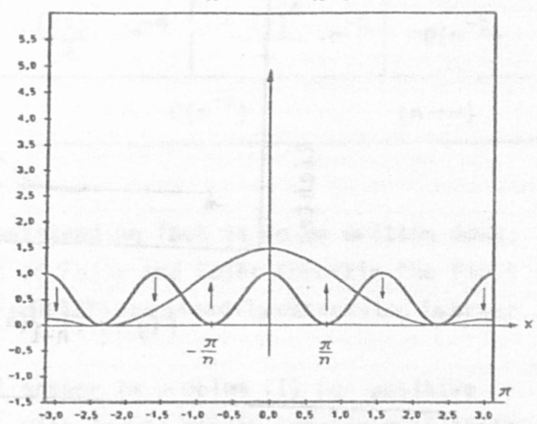


Fig.5: $\bar{F}^{[1]}_{n-2}(x)$, n=4

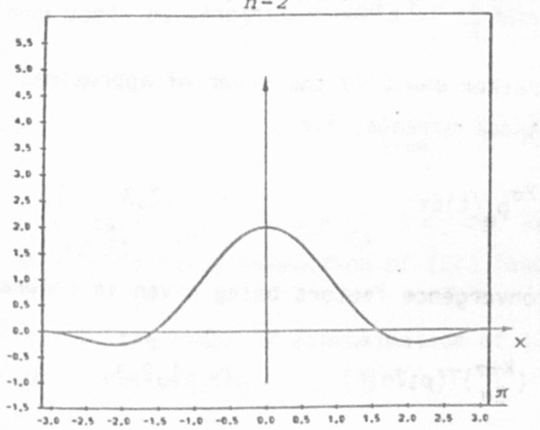
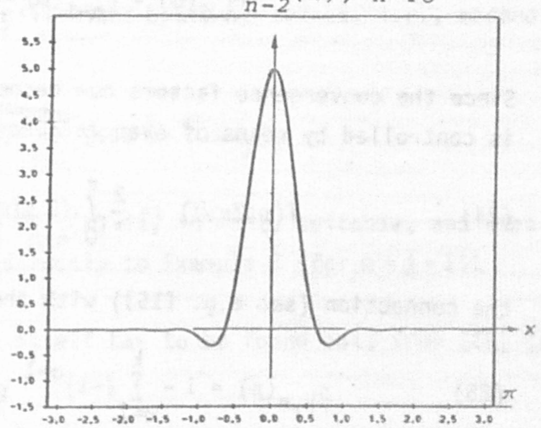


Fig.6: $\bar{F}^{[1]}_{n-2}(x)$, n=10



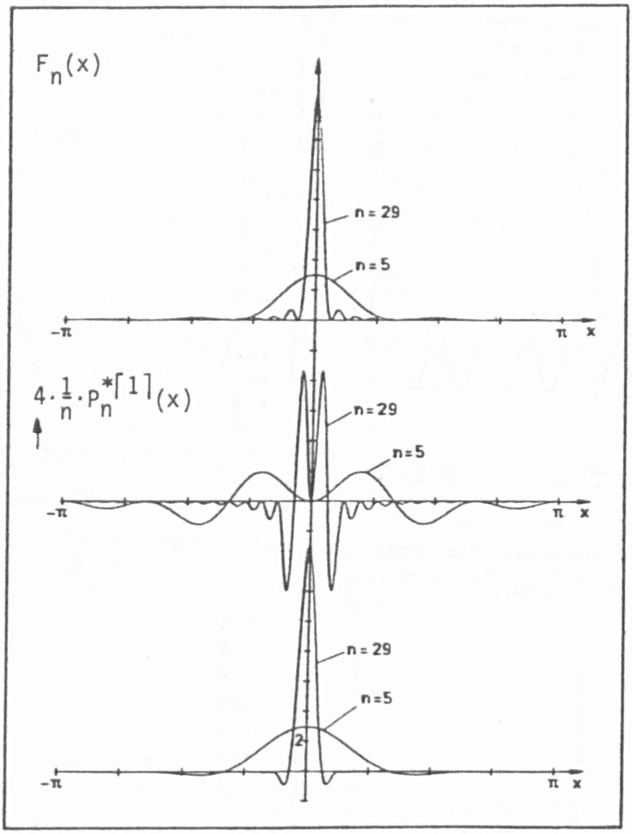


Fig.7: $F_{n-1}^{[1]}(x) = F_n(x) + \frac{1}{n} p_n^{*[1]}(x)$

$$\overline{F}^*_{n-3}(0) = F_{n-1}(0) \frac{2\cos^2\frac{\pi}{n}}{3-2\sin^2\frac{\pi}{n}} \simeq \frac{2}{3} F_{n-1}(0) = \frac{n}{3}, \qquad n \to \infty.$$

Since the convergence factors now become rather unwieldy the order of approximation is controlled by means of even *trigonometrical moments*, i.e.,

$$\text{(24)} \qquad T(p;2\sigma;n) := \frac{2}{\pi}\int_0^{\pi} \left(2\sin\frac{t}{2}\right)^{2\sigma} p_n(t)dt \qquad (\sigma = 1,2,3,\ldots),$$

the connection (see e.g. [15]) with the convergence factors being given in general by

$$\text{(25)} \qquad \rho_{k,n}(p) = 1 - \sum_{\sigma=1}^{k} (-1)^{\sigma+1} \frac{k}{k+\sigma} \binom{k+\sigma}{2\sigma} T(p;2\sigma;n) \qquad (k = 1,2,3,\ldots);$$

for instance, in the simplest case ($k=1$; giving the saturation order)

$$\rho_{1,n}(p) = 1 - \tfrac{1}{2}T(p;2;n) \Leftrightarrow T(p;2;n) = 2\big(1-\rho_{1,n}(p)\big). \tag{26}$$

The moments (24) are evaluated by methods corresponding to those for the normalization factors. In order to classify the positive kernel (23) its first moments are included in the following table.

$T(p;2\sigma;n)$:				
2σ \ p	2	4	6	8,10,12,..
$F_{n-1} =$	$2\cdot n^{-1}$	$4\cdot n^{-1}$	$12\cdot n^{-1}$	$O(n^{-1})$
$\overline{F}^*_{n-3} \cong$	$\frac{(2\pi)^2}{3}\cdot n^{-2}$	$\frac{(2\pi)^4}{3}\cdot n^{-4}$	$\frac{4(2\pi)^4}{3}\cdot n^{-5}$	$O(n^{-5})$
$K_n \cong$	$\pi^2\cdot n^{-2}$	$O(n^{-3})$		$(n\to\infty)$

Since this table is self-evident, only the outstanding fact is to be written down: in comparison with the original kernels (10) of Fejêr and Fejêr-Korovkin the first *three* trigonometrical moments of the new kernel (23) are steadily decreasing in order.

It is this decisive feature giving a partial answer to problem (I) for positive kernels. It should be indicated that kernels with "good" growth behaviour of their moments may, in turn, be used in order to construct kernels of finite oscillations. Up to now, in contrast to the procedure of problem (II) the *inverse* construction has been used: building up kernels of finite oscillations by *adding* zeroes, i.e., according to

$$t_{n+m}(x) := p_n(x) \prod_{j=1}^{m} (\cos x - \cos\alpha_j) \tag{27}$$

for suitable $p_n(x) \geq 0$ and α_j, $1 \leq j \leq m$. Here $\overline{F}^*_{n-3}(x)$ is, in fact, suitable, and that for $S^{(2,3)}$: the construction of (27) leads directly to Example 2 (for $m=j=1$)!

Finally, the order of approximation of (23) itself has to be found out; from (25) and (26) it follows that

$$\lim_{n\to\infty} \frac{1-\rho^*_{2,n-3}(F)}{1-\rho_{1,n-3}(F)} = 4 \quad \Leftrightarrow \quad \lim_{n\to\infty} \frac{1-\rho^*_{k,n-3}(F)}{1-\rho_{1,n-3}(F)} = k^2 \quad (k=3,4,5,\ldots)$$

where the latter is due to a well-known equivalence theorem for convergence factors (see e.g. [6, p. 450], [8, p. 76], [15]); this, on the other hand, implies the saturation limit

$$\lim_{n\to\infty} n^2\left(1-\rho^*_{k,n-3}(F)\right) = \frac{2\pi^2}{3} k^2 \qquad (k=1,2,3\ldots)$$

such that, comparing with (12), the new apid (23) is close to the classical Fejêr-Korovkin kernel.

4. Concluding remarks. It should be mentioned that some computer experiments have been (and will be) of great advantage: a series of figures as the output of the given procedure enabled an easier and effective analytical description of various, partly most surprising phaenomena which occured during these investigations.

Parts of this paper are based upon [13], containing also, in addition to the restricted selection of examples presented here, a series of general theorems (and proofs) which together with some further developments will be published elsewhere.

Finally, it must be emphasized that the procedure of removing zeroes and renormalizing has been used in case of (optimal, positive) algebraic approximation in previous papers, see e.g. [7], [3], [4] as well as [11] where the way leads from Čebyšev polynomials to the Fejêr-Korovkin kernel.

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