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APPROXIMATION BY GENERALIZED SAMPLING SERIES

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1. Introduction. The well known Whittaker - Kotelnikov - Shannon sampling theorem states that any signal f which is bandlimited to $[-\pi\sigma,\pi\sigma]$ can be completely reconstructed from its sample values $f(k/\sigma)$, equally spaced apart on the real axis $\mathbb{R}$, in terms of

$$(1.1)\qquad f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{\sigma}\right) \frac{\sin\pi(\sigma t-k)}{\pi(\sigma t-k)} \qquad (t\in\mathbb{R}).$$

If f is not bandlimited but has an absolutely integrable Fourier transform $f^{\wedge}(v) := (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} f(u)e^{-ivu}\,du$, in notation $f^{\wedge}\in L(\mathbb{R})$, then (1.1) holds in the limit for $\sigma\to\infty$, namely

$$(1.2)\qquad f(t) := \lim_{\sigma\to\infty} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{\sigma}\right) \frac{\sin\pi(\sigma t-k)}{\pi(\sigma t-k)} \qquad (t\in\mathbb{R}).$$

Instead of assuming $f^{\wedge}\in L(\mathbb{R})$ one may also suppose that f satisfies a Dini - Lipschitz condition; but it can be shown that just continuity of f is not sufficient for (1.2) even if f has compact support such that the infinite series exists for all t and σ. For these facts see e.g. [1; 2; 3; 8; 10; 11].

The aim of this paper is to replace the kernel $\sin\pi t/\pi t$ in (1.2) by another more suitable function φ such that the (generalized) sampling series

$$(1.3)\qquad (\Phi_\sigma f)(t) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{\sigma}\right) \varphi(\sigma t-k)$$

exists for all $f\in C(\mathbb{R}), t\in\mathbb{R}, \sigma>0$, and converges for $\sigma\to\infty$ uniformly towards f(t). Here $C(\mathbb{R})$ is the space of all complex - valued, uniformly continuous and bounded functions f defined on $\mathbb{R}$, endowed with the supremum norm $\|f\|_C$. The subset [of] those $f\in C(\mathbb{R})$ satisfying $\lim_{t\to\pm\infty} f(t)=0$ is denoted by $C_0(\mathbb{R})$. As usual, $\mathbb{N}$ and $\mathbb{Z}$ are the sets of all naturals and integers, respectively.

1) Supported by the Stiftung Volkswagenwerk

2. Admissible kernels for generalized sampling series. In this section we give necessary and sufficient conditions upon φ such that

$$(2.1) \qquad \lim_{\sigma\to\infty} \|\Phi_\sigma f - f\|_C = 0 \qquad (f \in C(\mathbb{R})) .$$

In order to ensure that the infinite series (1.3) itself is convergent for every $f \in C(\mathbb{R}), t \in \mathbb{R}, \sigma > 0$, and defines a continuous function of t, φ is required to satisfy

$$(2.2) \qquad \varphi \in C(\mathbb{R}), \quad \sum_{k=-\infty}^{\infty} |\varphi(u-k)| < \infty \quad \text{uniformly for } u \in [0,1] .$$

Lemma 1: Let φ satisfy (2.2). One has

a) $\varphi \in L(\mathbb{R}) \cap C_0(\mathbb{R})$.

b) The series in (2.2) converges absolutely for all $u \in \mathbb{R}$, the absolute convergence being uniform on each compact subinterval of $\mathbb{R}$, and $m_0(\varphi) := \|\sum_{k=-\infty}^{\infty}|\varphi(\cdot-k)|\|_C < \infty$.

c) Uniformly for all $u \in \mathbb{R}$,

$$\lim_{R\to\infty} \sum_{|k-u|>R} |\varphi(u-k)| = 0 .$$

Proof. Since

$$(2.3) \qquad \int_{-\infty}^{\infty} |\varphi(u)|\, du = \sum_{k=-\infty}^{\infty} \int_0^1 |\varphi(u-k)|\, du = \int_0^1 \sum_{k=-\infty}^{\infty} |\varphi(u-k)|\, du < \infty ,$$

it follows that $\varphi \in L(\mathbb{R})$. Next consider the convolution integral $(I_\rho\varphi)(u) := \int_{-\infty}^{\infty} \varphi(u-x)\rho\, F(\rho x)\, dx$, where $F(x) := (1/2)[\sin(\pi x/2)/(\pi x/2)]^2$ is Fejér's kernel. It is well known that $I_\rho\varphi \in C_0(\mathbb{R})$ for each $\rho > 0$ and that $\lim_{\rho\to\infty}\|I_\rho\varphi-\varphi\|_C = 0$ (cf. [5, Prop. 0.2.1, Cor. 3.1.10]). Hence φ belongs $C_0(\mathbb{R})$, too. This proves part a), and part b) is obvious in view of the fact that the series in (2.2) defines a function with period one. Finally, let [u] denote the largest integer $\leq u$ and set $u_0 = u - [u]$. Then $0 \leq u_0 \leq 1$ and

$$\sum_{|k-u|\geq R} |\varphi(u-k)| = \sum_{|k-u_0|\geq R} |\varphi(u_0-k)| \leq \sup_{0\leq u_0\leq 1} \sum_{|k|\geq R-1} |\varphi(u_0-k)| ,$$

which tends to zero for $\sigma\to\infty$ by (2.2). □

As an immediate consequence one has

Corollary 1: The operators Φ_σ, defined for $\sigma > 0$ by (1.3), are bounded linear operators from $C(\mathbb{R})$ into ifself, and satisfy

(2.4) $$\|\Phi_\sigma\| = m_0(\varphi) \qquad (\sigma > 0).$$

Proof. Fix $\sigma > 0$ and choose $\varepsilon > 0$ arbitrary. By La. 1 c) one can find $N \in \mathbb{N}$ such that

$$\sum_{|k-\sigma t|\geq N-1} |\varphi(\sigma t-k)| < \frac{\varepsilon}{3} \qquad (t \in \mathbb{R}).$$

Furthermore, by the uniform continuity of φ, there exists $0 < \delta \leq 1/\sigma$ such that $\|\varphi(\cdot+y)-\varphi(\cdot)\|_C < \varepsilon/(6N+3)$ for all $|y| < \delta\sigma$. Now for $|h| < \delta$

$$|(\Phi_\sigma f)(t+h) - (\Phi_\sigma f)(t)| \leq \sum_{|k-\sigma t|<N} |f(\tfrac{k}{\sigma})|\,|\varphi(\sigma t+\sigma h-k) - \varphi(\sigma t-k)|$$

$$+ \sum_{|k-\sigma t|\geq N} |f(\tfrac{k}{\sigma})|\,|\varphi(\sigma t+\sigma h-k| + \sum_{|k-\sigma t|\geq N} |f(\tfrac{k}{\sigma})|\,|\varphi(\sigma t-k)|$$

$$\leq \|f\|_C \{(2N+1)\frac{\varepsilon}{6N+3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\} = \varepsilon\|f\|_C,$$

noting that the first series contains at most 2N+1 terms and that the second series can be estimated by

$$\sum_{|k-\sigma t|\geq N} |f(\tfrac{k}{\sigma})|\,|\varphi(\sigma t+\sigma h-k)| \leq \|f\|_C \sum_{|k-\sigma t-\sigma h|\geq N-1} |\varphi(\sigma t+\sigma h-k)|$$

since $|k-\sigma t| \leq |k-\sigma t-\sigma h| + 1$. This proves the uniform continuity of $\Phi_\sigma f$. The boundedness follows from La. 1 b) by

$$|(\Phi_\sigma f)(t)| \leq \|f\|_C\, m_0(\varphi) \qquad (t \in \mathbb{R}),$$

giving also the inequality "$\leq$" in (2.4). For the converse inequality let u_0 be a point with $\sum_{k=-\infty}^{\infty}|\varphi(u_0-k)| = m_0(\varphi)$. Then take any $g \in C(\mathbb{R})$ with $\|g\|_C = 1$ and $g(k/\sigma) = \operatorname{sgn}\varphi(u_0-k)$. This yields

$$\|\Phi_\sigma g\|_C \geq (\Phi_\sigma g)(u_0) = \sum_{k=-\infty}^{\infty} [\operatorname{sgn}\varphi(u_0-k)]\,\varphi(u_0-k) = m_0(\varphi)$$

and completes the proof. □

The main result of this section now reads

Theorem 1: If φ satisfies (2.2), then each of the following assertions is equivalent to (2.1):

$$(2.5)\qquad \sum_{k=-\infty}^{\infty} \varphi(u-k) = 1 \qquad (u \in \mathbb{R}),$$

$$(2.6)\qquad \varphi^{\wedge}(2k\pi) = \begin{cases} \frac{1}{\sqrt{2\pi}}, & k = 0 \\ 0, & k \in \mathbb{Z}\setminus\{0\}. \end{cases}$$

Proof. Let us first show that (2.5) and (2.6) are equivalent. The series in (2.5) defines a continuous function with period one, and by Poisson's summation formula (cf. [5,p.201]) its Fourier series is given by

$$(2.7)\qquad \sum_{k=-\infty}^{\infty} \varphi(u-k) \sim \sum_{k=-\infty}^{\infty} \sqrt{2\pi}\, \varphi^{\wedge}(2k\pi)e^{i2k\pi u}$$

Now, if (2.6) is satisfied, then the series on the right side of (2.7) reduces to the term for $k=0$, which is equal to 1, and hence represents the function on the left. This gives (2.6) $\Rightarrow$ (2.5). Conversely, if (2.5) holds, then the Fourier series of the function $\sum_{k=-\infty}^{\infty} \varphi(u-k)$ has only one nonzero term, namely that for $k=0$ which is equal to 1; comparing this with (2.7) gives (2.6).

It remains to show that (2.5) is equivalent to (2.1). It follows from (2.5) that for $f \in C(\mathbb{R})$ and any $\delta > 0$

$$|(\Phi_\sigma f)(t)-f(t)| \leq \Big(\sum_{|\frac{k}{\sigma}-t|<\delta} + \sum_{|\frac{k}{\sigma}-t|\geq\delta} \Big) |f(\tfrac{k}{\sigma})-f(t)|\,|\varphi(\sigma t-k)|$$

$$\leq m_0(\varphi) \sup_{|h|\leq\delta} \|f(\cdot+h)-f(\cdot)\|_C + 2\|f\|_C \sum_{|k-\sigma t|\geq\sigma\delta} |\varphi(\sigma t-k)|\,.$$

By choosing $\delta > 0$ appropriately the first term on the right can be made arbitrarily small, f being uniformly continuous, whereas the second tends uniformly to zero for $\sigma \to \infty$ by La. 1 c). On the other hand, if (2.1) is valid, then one has for $f(t) \equiv 1$ and $t=1$ that $\lim_{\sigma\to\infty} \sum_{k=-\infty}^{\infty} \varphi(\sigma-k) = 1$. Since this series defines a periodic function it must be identically equal to 1, which is (2.5). ◻

In the following we call a function φ, which satisfies (2.2) and (2.5)(or(2.6)), an approximate identity.

Note that $\Phi_\sigma f$ can be regarded as a discrete version of the convolution integral

$$(J_\sigma f)(t) := \int_{-\infty}^{\infty} f(u)\, \sigma\varphi(\sigma(t-u))\, du\,.$$

For those integrals one has

$$(2.8)\qquad \lim_{\sigma\to\infty} \|J_\sigma f-f\|_C = 0 \quad (f\in C(\mathbb{R})) \iff \int_{-\infty}^{\infty} \varphi(u)\,du = 1$$

This corresponds to the equivalence of (2.1) and (2.5). But the right hand side of (2.8) may also be read as $\varphi^\wedge(0) = 1/\sqrt{2\pi}$. Then (2.8) corresponds to (2.1) $\iff$ (2.6). However, in case of the integrals $J_\sigma f$ there is only one condition upon the Fourier transform $\varphi^\wedge$ at $v=0$, whereas in case of the series $\Phi_\sigma f$ one has a countable number of conditions upon $\varphi^\wedge$, namely at all points $2k\pi$, $k\in\mathbb{Z}$.

3. Order of approximation. In this section the rate of convergence in (2.1) is to be investigated. It will turn out that this rate is intimately connected with the moments

$$\sum_{k=-\infty}^{\infty} (u-k)^j\,\varphi(u-k) \qquad (u\in\mathbb{R};\, j\in\mathbb{N}),$$

as well as with the multiplicity of the zeros of the Fourier transform at $2k\pi$.

Let us define the αth absolute moment of φ by

$$m_\alpha(\varphi) := \sup_{u\in\mathbb{R}} \sum_{k=-\infty}^{\infty} |u-k|^\alpha |\varphi(u-k)| \qquad (\alpha \geqslant 0).$$

For $a=0$ this definition coincides with that given in La. 1 b). It is easy to see that $m_\alpha(\varphi)<\infty$ implies $m_\beta(\varphi)<\infty$ for any $0\leqslant\beta<\alpha$ together with $\int_{-\infty}^{\infty}|u|^\alpha|\varphi(u)|\,du<\infty$ (cf. (2.3)). From the latter it follows that the Fourier transform $\varphi^\wedge$ has a continuous derivative of any order $j\in\mathbb{N}$ with $j\leqslant\alpha$. A sufficient condition for $m_\alpha(\varphi)<\infty$ is $|\varphi(u)| = \mathcal{O}(|u|^{-\alpha-\gamma})$, $u\to\pm\infty$, for some $\gamma>1$.

As a measure of smoothness for the functions to be approximated we use the modulus of continuity and the associated Lipschitz class

$$\omega(f;\delta) := \sup_{|h|\leqslant\delta} \|f(\cdot+h)-f(\cdot)\|_C \qquad (f\in C(\mathbb{R});\, \delta>0),$$

$$\mathrm{Lip}\,\alpha := \{f\in C(\mathbb{R});\, \omega(f;\delta) = \mathcal{O}(\delta^\alpha),\ \delta\to 0+\} \qquad (\alpha>0).$$

Theorem 2: Let φ be an approximate identity.

a) If $m_\alpha(\varphi)<\infty$ for some $0<\alpha\leqslant 1$, then

$$f\in\mathrm{Lip}\,\alpha \Rightarrow \|\Phi_\sigma f-f\|_C = \mathcal{O}(\sigma^{-\alpha}) \qquad (\sigma\to\infty).$$

b) If $m_1(\varphi)\leqslant\infty$, then

(3.1) $$\|\Phi_\sigma f-f\|_C \leq M\,\omega(f;\sigma^{-1}) \qquad (f\in C(\mathbb{R});\sigma>0),$$

the constant M depending only on φ.

Proof. For $f\in \mathrm{Lip}\,\alpha$ there exists a constant L such that $|f(k/\sigma)-f(t)|\leq L|k/\sigma-t|^\alpha$, and so

(3.2) $$|(\Phi_\sigma f)(t)-f(t)| \leq \sum_{k=-\infty}^{\infty} |f(\tfrac{k}{\sigma})-f(t)|\,|\varphi(\sigma t-k)|$$

$$\leq L\sum_{k=-\infty}^{\infty} |\tfrac{k}{\sigma}-t|^\alpha|\varphi(\sigma t-k)| \leq Lm_\alpha(\varphi)\sigma^{-\alpha}.$$

Concerning b), it suffices to establish the Jackson-type inequality

(3.3) $$\|\Phi_\sigma f-f\|_C \leq M_1\|f^{(1)}\|_C\,\sigma^{-1} \qquad (f\in C^{(1)}(\mathbb{R});\sigma>0),$$

$C^{(s)}(\mathbb{R}) := \{f\in C(\mathbb{R});f^{(s)}\in C(\mathbb{R})\}$. Assertion (3.1) then follows by standart arguments. But inequality (3.3) can be easily deduced from the estimate $|f(k/\sigma)-f(t)|\leq\|f^{(1)}\|_C\,|k/\sigma-t|$. □

The best possible order which can be obtained from Thm. 2 is $O(\sigma^{-1})$ unless $f=$ constant. It might be expected that in case $m_2(\varphi)<\infty$ one would obtain approximation orders up to $O(\sigma^{-2})$, at least for even kernels (cf. [5, Prop. 3.4.1]). But we will see in the following that for kernels with finite second absolute moment a better order than $O(\sigma^{-1})$ can be achieved if and only if some additional assumptions are satisfied. We need the following auxiliary lemma.

Lemma 2: Let φ be an approximate identity with $m_r(\varphi)<\infty$ for some $r\in\mathbb{N}\setminus\{1\}$, and

$$\|\Phi_\sigma f-f\|_C = O(\sigma^{-r}) \qquad (f\in C^{(r)}(\mathbb{R});\sigma\to\infty).$$

Then one has

(3.4) $$|\sum_{k=-\infty}^{\infty}[k^j-\sigma^j]\varphi(\sigma-k)| = O(\sigma^{-r+j}) \qquad (\sigma\to\infty;j=1,2,\dots r-1).$$

Proof. Let $f_j(u)$ be a function in $C^{(r)}(\mathbb{R})$ which is equal to u for $|u-1|\leq 1$. Then

(3.5) $$\sigma^{-j}|\sum_{k=-\infty}^{\infty}[k^j-\sigma^j]\,\varphi(\sigma-k)|$$

$$\leq \sum_{k=-\infty}^{\infty} |(\tfrac{k}{\sigma})^j - f_j(\tfrac{k}{\sigma})||\varphi(\sigma-k)| + |(\Phi_\sigma f_j)(1) - 1|$$

$$= \sum_{|k-\sigma|>\sigma} |(\tfrac{k}{\sigma})^j - f_j(\tfrac{k}{\sigma})||\varphi(\sigma-k)| + O(\sigma^{-r}) \qquad (\sigma\to\infty).$$

Using now the assumption that the absolute moment of order r is finite and that f_j is bounded, the latter series can be estimated by

$$(3.6) \qquad \sum_{|k-\sigma|>\sigma} |(\tfrac{k}{\sigma})^j - f_j(\tfrac{k}{\sigma})||\varphi(\sigma-k)| \leq \sigma^{-j} \sum_{|k-\sigma|>\sigma} \frac{|k|^j}{|\sigma-k|^r} |\sigma-k|^r |\varphi(\sigma-k)|$$

$$+ \|f_j\|_C \sum_{|k-\sigma|>\sigma} \frac{1}{|\sigma-k|^r} |\sigma-k|^r |\varphi(\sigma-k)|$$

$$\leq \sigma^{-r}\{ \sum_{|k-\sigma|>\sigma} \frac{|k|^j}{|\sigma-k|^j} |\sigma-k|^r |\varphi(\sigma-k)| + \|f_j\|_C m_r(\varphi)\} .$$

Finally, since $|k|/|\sigma-k| \leq 2$ for $|k-\sigma| > 1$, it follows that

$$\sum_{|k-\sigma|>\sigma} |(\tfrac{k}{\sigma})^j - f_j(\tfrac{k}{\sigma})||\varphi(\sigma-k)| \leq \sigma^{-r}(2 + \|f_j\|_C)\, m_r(\varphi),$$

and inserting this into (3.5) gives (3.4). □

We are now in a stage to prove our main result concerning the rate of convergence in (2.1). It is completely analogous to Thm. 1.

Theorem 3: If φ is an approximate identity with $m_r(\varphi) < \infty$ for some $r \in \mathbb{N}\setminus\{1\}$, then the following assertions are equivalent:

(i) $\quad \|\Phi_\sigma f - f\|_C \leq M \|f^{(r)}\|_C\, \sigma^{-r} \qquad (f \in C^{(r)}(\mathbb{R});\ \sigma > 0)$,

(ii) $\quad \sum_{k=-\infty}^{\infty} (u-k)^j \varphi(u-k) = 0 \qquad (u \in \mathbb{R};\ j = 1,2,\ldots,r-1)$,

(iii) $\quad [\varphi^\wedge]^{(j)}(2k\pi) = 0 \qquad (k \in \mathbb{Z};\ j = 1,2,\ldots,r-1)$.

Proof. Assume that (i) holds. Then by the binomial formula

$$(\sigma-k)^j = \sum_{\nu=1}^{j} \binom{j}{\nu}(-1)^\nu \sigma^j[(\tfrac{k}{\sigma})^\nu - 1]$$

one obtains

$$\sum_{k=-\infty}^{\infty} (\sigma-k)^j \varphi(\sigma-k) = \sigma^j \sum_{\nu=1}^{j} \binom{j}{\nu}(-1)^\nu \sum_{k=-\infty}^{\infty} [(\tfrac{k}{\sigma})^\nu - 1]\, \varphi(\sigma-k),$$

and since the infinite series on the right are of order $O(\sigma^{-r})$ by La.2, it follows that

$$\lim_{\sigma\to\infty} \sum_{k=-\infty}^{\infty} (\sigma-k)^j \varphi(\sigma-k) = 0. \tag{3.7}$$

This implies (ii) as in the proof of Thm. 1, (2.1) ⇒ (2.5).

As to the converse direction, one has by Taylor's formula

$$|(\Phi_\sigma f)(t)-f(t)| = \sum_{j=1}^{r-1} \frac{f^{(j)}(t)}{j!} \sum_{k=-\infty}^{\infty} (\tfrac{k}{\sigma}-t)^j \varphi(\sigma t-k)$$

$$+ \frac{1}{r!} \sum_{k=-\infty}^{\infty} f^{(r)}(\xi)(\tfrac{k}{\sigma}-t)^r \varphi(\sigma t-k),$$

where ξ depends on k,σ and t. By (ii) the double series vanishes, and one can estimate the approximation error by

$$\|\Phi_\sigma f-f\|_C \le \frac{1}{r!} \|f^{(r)}\|_C\, \sigma^{-r} m_r(\varphi).$$

This is (i). The equivalence of (ii) and (iii) can be proved in the same manner as (2.5) ⇔ (2.6), using the fact that $[u^j\varphi(u)]^\wedge(v) = (-i)^j [\varphi^\wedge]^{(j)}(v)$ (cf. [Prop. 5.1.17]), i.e.,

$$\sum_{k=-\infty}^{\infty} (u-k)^j \varphi(u-k) \sim \sum_{k=-\infty}^{\infty} (-i)^j \sqrt{2\pi}\, [\varphi^\wedge]^{(j)}(2k\pi) e^{i2k\pi u}. \qquad \square$$

It should be mentioned, that the condition $m_r(\varphi) < \infty$ in Thm. 3 is essential. In [6] the kernel $\sin \pi t \sin(\pi t/2)/(\pi^2 t^2/2)$ was considered. Although it does not even have a finite absolute moment of order 1, the associated sampling sum approximates with an arbitrarily fast rate if the function f is smooth enough. On the other hand, if φ has compact support, then $m_r(\varphi) < \infty$ for all $r \ge 2$, and $\varphi^\wedge$ is an entire function. This means in particular that there exists some r_0 such that (i) holds for $r = r_0$ but not for $r = r_0+1$. Indeed, if (i) would be were valid for each $r \ge 2$ then $\varphi^\wedge$ would have zeros of infinite order, implying $\varphi^\wedge(v) \equiv 0$. This would contradict the assumption $\varphi^\wedge(0) = 1$.

As a corollary one easily deduces the following Jackson-type theorems.

Corollary 2: Let φ be an approximate identity with $m_r(\varphi) < \infty$ for some $r \in \mathbb{N}\setminus\{1\}$ satisfying (ii) or (iii) of Thm 3.

a) For each $j=0,1,\dots,r-1$ there holds

$$\|\Phi_\sigma f-f\|_C \le M\,\sigma^{-j}\omega(f^{(j)};\,\sigma^{-1}) \qquad (f\in C^{(j)}(\mathbb{R});\,\sigma>0)\,.$$

b) If $f^{(j)}\in \mathrm{Lip}\,\alpha$ for some $j=0,1,\dots,r-1$ and some $0<\alpha\le 1$, then

$$\|\Phi_\sigma f-f\|_C = \mathcal{O}(\sigma^{-j-\alpha}) \qquad (\sigma\to\infty)\,.$$

We conclude this section by showing that the orders in Cor. 2 b) cannot be improved, at least for $0<\alpha<1$.

Lemma 3: Let φ be given as in Cor. 2. For each $j=0,1,\dots r-1$ and each $0<\alpha<1$ there exists a function f_o with $f_o^{(j)}\in\mathrm{Lip}\,\alpha$ such that

$$\|\Phi_\sigma f_o-f_o\|_C \ne o(\sigma^{-j-\alpha}) \qquad (\sigma\to\infty)\,.$$

Proof. According to a result of Dickmeis and Nessel [7] it is enough to show that there exists a family $\{g_\sigma\}_{\sigma>0}\subset C^{(r)}(\mathbb{R})$ having the properties

$$\text{(3.8)} \qquad \|g_\sigma\|_C \le M_1\,, \quad \|g^{(r)}\|_C \le M_2\,\sigma^r \qquad (\sigma>0)\,,$$

$$\text{(3.9)} \qquad \|\Phi_\sigma g_\sigma - g_\sigma\|_C \ge M_3>0 \qquad (\sigma>0)\,.$$

Taking $g_\sigma(t)=\cos 2\pi\sigma t$, then (3.8) is obviously satisfied. Moreover, it follows from

$$\sum_{k=-\infty}^{\infty}\cos 2\pi\sigma\frac{k}{\sigma}\,\varphi(\sigma t-k)-\cos 2\pi\sigma t = \sum_{k=-\infty}^{\infty}\varphi(\sigma t-k)-\cos 2\pi\sigma t = 1-\cos 2\pi\sigma t$$

that (3.9) holds with $M_3=2$. □

4. Applications.

Here we will give two typical examples of kernels φ. First let $\varphi=F$ be Fejér's kernel already used in the proof of La. 1. In this case the sampling series reads

$$\text{(4.1)} \qquad (\Phi_\sigma^F f)(t)=\frac{1}{2}\sum_{k=-\infty}^{\infty} f\left(\frac{k}{\sigma}\right)\left(\frac{\sin[\pi(\sigma t-k)/2]}{\pi(\sigma t-k)/2}\right)^2 .$$

The Fourier transform is given by

$$F^{\wedge}(v) = \begin{cases} 1 - \frac{|v|}{\pi} , & |v| \leq \pi \\ 0 , & |v| > \pi , \end{cases}$$

which immediately shows that F is an approximate identity. Concerning the order of approximation one may apply Thm. 2. This yields

Corollary 3: If Φ_σ^F is defined by (4.1), then (2.1) holds for $\Phi_\sigma = \Phi_\sigma^F$, and one has for each $0 < \alpha < 1$

$$f \in \mathrm{Lip}\,\alpha \Rightarrow \|\Phi_\sigma^F f - f\|_C = O(\sigma^{-\alpha}) \qquad (\sigma \to \infty) .$$

For the case $\alpha = 1$ see [6; 9].

Our next example is $\varphi = M$, defined by

$$(4.2) \qquad M(t) = \begin{cases} 1 - |t| , & |t| \leq 1 \\ 0 , & |t| > 1 , \end{cases} \qquad \widehat{M}(v) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sin v/2}{v/2}\right)^2$$

M is the B-spline of degree 1, and $\Phi_\sigma^M f$ is the linear spline which interpolates f at the points k/σ. From Thm. 1 and Thm. 3 one obtains

Corollary 4: Let $\Phi_\sigma^M f$ be the generalized sampling series with kernel M given by (4.2). Then (2.1) holds for $\Phi_\sigma = \Phi_\sigma^M$ and one has for $j = 0,1$ and $0 < \alpha \leq 1$

$$f^{(j)} \in \mathrm{Lip}\,\alpha \Rightarrow \|\Phi_\sigma^M f - f\|_C \leq M \sigma^{-1} \omega(f^{(j)}; \sigma^{-1}) \qquad (\sigma > 0).$$

For further examples see [4; 6; 8; 9] .

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