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ZERO-FREE REGIONS FOR POLYNOMIALS WITH APPLICATIONS TO PADÉ-APPROXIMANTS

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1. Introduction. The object of this paper is to determine zero-free regions for polynomials $q_n(z)$ satisfying

$$(0)\quad q_n(z) = (z+b_n)q_{n-1}(z)-a_n z\, q_{n-2}(z) \text{ for } n \in \mathbb{N}, \text{ where } q_0 = 1,\ q_{-1} = 0\ ,\ a_{n+1},\ b_n > 0 \text{ for } n \in \mathbb{N}\ .$$

2. The basic continued fraction method. We associate $q_n(z)$ with the continued fraction

$$w_n(z) = \frac{p_n(z)}{q_n(z)} = \frac{1}{z+b_1} - \frac{a_2 z}{z+b_2} - \cdots - \frac{a_n z}{z+b_n}$$

and observe that $q_n(z) = 0$ iff $w_n(z) = \infty$. After applying an equivalence transformation to this continued fraction we obtain for $z \neq 0$

$$w_n(z) = \frac{1/z}{1+b_1/z} - \frac{a_2/z}{1+b_2/z} - \cdots - \frac{a_n/z}{1+b_n/z}$$

and, hence, $w_n(z) = s_1 \circ s_2 \circ \cdots \circ s_n(0)$, where $s_1(u) := \dfrac{1/z}{(1+b_1/z)+u}$, and

$$s_n(u) := \frac{-a_n/z}{(1+b_n/z)+u} \quad \text{for } n \geq 2 \text{ and } u \in \overline{\mathbb{C}} = \mathbb{C}\cup\{\infty\}\ .$$

We want to find conditions on $\alpha_{n+1} := -a_{n+1}/z$ and $\beta_n := 1+b_n/z$, $n \in \mathbb{N}$, which imply $w_n(z) \neq \infty$ for $1 \leq n \leq N$, where $N \geq 2$. Such conditions are obtained by choosing closed half-planes $H_n \subset \mathbb{C}$, $1 \leq n \leq N$, such that $s_n(H_n)$ is a finite disk for $1 \leq n \leq N$ and $s_n(H_n) \subset H_{n-1}$ holds for $2 \leq n \leq N$. Then $0 \in H_N$ implies $w_N(z) \in s_1(H_1)$ and, hence, $w_N(z) \neq \infty$. This method is part of well-known convergence considerations for continued fractions.

Choosing especially $H_n = H := \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta + d \geq 0\} \cup \{\infty\}$, where $d \geq 0$, i.e. $0 \in H$, then $s_n(H)$ is a finite disk iff $-\beta_n \notin H$, i.e. $\operatorname{Re}\beta_n > d$ for $1 \leq n \leq N$. $s_n(H) \subset H$ holds iff $2d(\operatorname{Re}\beta_n - d) \geq |\alpha_n| - \operatorname{Re}\alpha_n$, $2 \leq n \leq N$. We thus obtain

THEOREM 1. If for $N \geq 2$ $z = re^{i\vartheta}$, $r = r(\vartheta) > 0$, satisfies

(1) $b_n \cos\vartheta + r(\vartheta)(1-d) > 0$, $1 \leq n \leq N$,

(2) $2d(b_n \cos\vartheta + r(\vartheta)(1-d)) \geq a_n(1+\cos\vartheta)$, $2 \leq n \leq N$,

where $d = d(z) \geq 0$, then $q_n(z) \neq 0$ for $1 \leq n \leq N$.

3. The main results. Especially, for $d = d(z) = (1 + \cos\vartheta)/2$ we obtain

THEOREM 2. If for $N \geq 2$ $B_N := \max_{1 \leq n \leq N} b_n$,

$Q_N := \max_{2 \leq n \leq N} (a_n/b_n)$, then $q_n(z) \neq 0$ for $1 \leq n \leq N$

and all $z = r e^{i\vartheta}$, $r = r(\vartheta) > 0$, satisfying

(3) $r(\vartheta) \geq 2B_N \left(\dfrac{Q_N - \cos\vartheta}{1 - \cos\vartheta} \right)$ $(r(0) = +\infty$ if $Q_N > 1)$.

Proof. If $0 < |\vartheta| \leq \pi$, then for $2 \leq n \leq N$ (3) implies $r(\vartheta)(1-\cos\vartheta) \geq 2 B_N (Q_N - \cos\vartheta) \geq 2 b_n((a_n/b_n) - \cos\vartheta) = 2(a_n - b_n \cos\vartheta)$. Using $1 - d = (1-\cos\vartheta)/2$, we obtain $b_n \cos\vartheta + r(\vartheta)(1-d) \geq a_n > 0$ for $2 \leq n \leq N$. Again (3) implies $r(\vartheta)(1-\cos\vartheta) \geq 2 b_1(Q_N - \cos\vartheta)$ and, hence, $b_1 \cos\vartheta + r(\vartheta)(1-d) \geq b_1 Q_N > 0$. Therefore (1) is satisfied. Multiplying $b_n \cos\vartheta + r(\vartheta)(1-d) \geq a_n$ by $d = (1 + \cos\vartheta)/2$ shows that (2) is satisfied. For $Q_N \leq 1$ and $\vartheta = 0$ (1) reduces to $b_n > 0$, $1 \leq n \leq N$, and (2) reduces to $2 b_n \geq 2 a_n$, $2 \leq n \leq N$, which is satisfied because of $a_n/b_n \leq Q_N \leq 1$. For $Q_N > 1$ and $\vartheta = 0$ $r(0) = +\infty$. Therefore Theorem 2 is a consequence of Theorem 1.

Next we use the following (see [1], [2], [4])

PARABOLA THEOREM (E.B. Saff, R.S. Varga). If $D_N := \min_{1 \leq n \leq N} (b_n - a_n) > 0$ (with $a_1 = 0$), then $q_n(z) \neq 0$ for $2 \leq n \leq N$ and all $z = r(\vartheta)e^{i\vartheta}$ satisfying $|z| \leq \operatorname{Re} z + 2 D_N$, i.e. $r(\vartheta)(1-\cos\vartheta) \leq 2 D_N$.

Observe that always $Q_N \leq 1 - D_N/B_N$ holds. Therefore, $D_N > 0$ implies $Q_N < 1$. Now the curves $r(\vartheta)(1-\cos\vartheta) = 2D_N$ and $r(\vartheta)(1-\cos\vartheta)=2B_N(Q_N-\cos\vartheta)$ intersect for $\vartheta=\vartheta_N$ given by $2D_N = 2B_N(Q_N-\cos\vartheta)$ i.e. $\vartheta_N = \arccos(Q_N-D_N/B_N)$. Then $r(\vartheta_N) = 2D_N/(1-Q_N+D_N/B_N)$. This leads us to

THEOREM 3. If for $N \geq 2$ $B_N := \max\limits_{1\leq n\leq N} b_n$,

$Q_N := \max\limits_{2\leq n\leq N} (a_n/b_n)$ and if $D_N := \min\limits_{1\leq n\leq N} (b_n - a_n) > 0$ $(a_1=0)$, then

$q_n(z) \neq 0$ for $1 \leq n \leq N$ and all $z = r\,e^{i\vartheta}$ satisfying

(4) $|\vartheta| \leq \vartheta_N = \arccos(Q_N-D_N/B_N)$ (where $Q_N-D_N/B_N \leq 1 - 2\,D_N/B_N$).

Proof. Assume that z satisfies $\cos\vartheta \geq Q_N-D_N/B_N$. Then $1-\cos\vartheta \leq 1-Q_N+D_N/B_N$. If $r = r(\vartheta) \leq r(\vartheta_N) = 2D_N/(1-Q_N+D_N/B_N)$, then $r(\vartheta)(1-\cos\vartheta) \leq 2D_N$ and, by the Parabola Theorem, $q_n(z) \neq 0$, $2 \leq n \leq N$.

If $r(\vartheta) \geq r(\vartheta_N)$, then $r(\vartheta_N) = 2B_N\left(\dfrac{Q_N - (Q_N-D_N/B_N)}{1 - (Q_N-D_N/B_N)}\right) \geq 2B_N\,\dfrac{Q_N-\cos\vartheta}{1-\cos\vartheta}$

because of $Q_N-D_N/B_N \leq \cos\vartheta$ and since $(Q_N-x)/(1-x)$ decreases when $x < 1$ increases. Hence, $q_n(z) \neq 0$, $1 \leq n \leq N$ by Theorem 2.
Since always $\cos\vartheta_N > -1$ the zero $-b_1$ of q_1 is not contained in (4).

REMARK. If $Q_N \leq Q$ and $D_N/B_N \geq A$ for all N, then $q_n(z) \neq 0$ for all $n \in \mathbb{N}$ and all $z = r\,e^{i\vartheta}$ satisfying $\cos\vartheta \geq Q - A$.

The following example shows that Theorem 3 is sharp for $N = 2$ in the sense that to every ε with $0 < \varepsilon < 1$ there is a polynomial $q_2(z)$ with zero $z = r\,e^{i\varphi}$ satisfying $\cos\varphi + \varepsilon = \cos\vartheta_2$. Assume that $b_1=b_2-a_2=D_2 > 0$. Then $B_2 = b_2$, $Q_2 = a_2/b_2$ and, hence, $Q_2 - D_2/B_2 = (a_2-D_2)/b_2 = 1 - 2D_2/b_2 = \cos\vartheta_2$. The zeros of $q_2(z) = z^2 + (b_1+b_2-a_2)\,z + b_1b_2 = z^2 + 2D_2\,z + D_2b_2$ are $-D_2 \pm i(D_2(b_2-D_2))^{1/2} = r\,e^{\pm i\varphi}$ with $\cos\varphi = -(D_2/b_2)^{1/2}$. Let η with $0 < \eta < 1$ be defined by $D_2 =: \eta^2\,b_2$. Then $\varepsilon - (D_2/b_2)^{1/2} = 1 - 2\,D_2/b_2$ holds iff $2\eta^2 - \eta + \varepsilon - 1 = 0$ holds. This yields $\eta = \eta(\varepsilon) = (1 + (1+8(1-\varepsilon))^{1/2})/4$. $\varepsilon \to 0$ implies $\eta \to 1$, $D_2 \to b_2$, $Q_2 \to 0$, and $\cos\vartheta_2 \to -1$.

4. Application to Padé-approximants. The preceding results can be applied immediately to the Padé-numerator $U_{m,n}(z)$ and the Padé-denominator $V_{m,n}(z)$ of $f(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$, satisfying $A_m^{(n)} > 0$ for, $m,n \geq 0$,

$$\text{where } A_m^{(n)} = \begin{vmatrix} c_m & c_{m-1} & \cdots & c_{m-n+1} \\ c_{m+1} & c_m & \cdots & c_{m-n+2} \\ \vdots & & & \\ c_{m+n-1} & \cdots & & c_m \end{vmatrix} \quad \text{for } m \geq 0 , n \geq 1$$

$A_m^{(0)} = 1$ and $c_{-\nu} = 0$ for $\nu \geq 1$.

U) For fixed $n \geq 0$ $q_m(z) := U_{m,n}(z) A_m^{(n)}/A_m^{(n+1)}$ satisfies (0) with

$a_{m+1} = A_{m+1}^{(n)} A_{m-1}^{(n+1)}/A_m^{(n)} A_m^{(n+1)}$, $b_m = A_{m-1}^{(n+1)} A_m^{(n)}/A_{m-1}^{(n)} A_m^{(n+1)}$, $m \geq 1$,

and for $M \geq 2$ one obtains $D_M = \min_{1 \leq m \leq M} A_m^{(n)} A_{m-1}^{(n+2)}/A_m^{(n+1)} A_{m-1}^{(n+1)} > 0$ and

$Q_M = \max_{2 \leq m \leq M} A_m^{(n+1)} A_{m-2}^{(n+1)} / (A_{m-1}^{(n+1)})^2 < 1$.

Especially, for $n = 0$ $b_m = a_{m+1} = c_{m-1}/c_m$, $m \geq 1$ and $U_{m,0}(z)$ is the m-th partial sum of $f(z)$.

Then $B_M = \max_{1 \leq m \leq M} c_{m-1}/c_m$, $D_M = \min_{1 \leq m \leq M} \left(\frac{c_{m-1}}{c_m} - \frac{c_{m-2}}{c_{m-1}} \right) > 0$

and $Q_M = \max_{2 \leq m \leq M} c_{m-2}\, c_m/c_{m-1}^2 < 1$.

V) For fixed $m \geq 0$ $q_n(z) := V_{m,n}(-z) A_m^{(n)}/A_{m+1}^{(n)}$ satisfies (0) with

$a_{n+1} = A_m^{(n+1)} A_{m+1}^{(n-1)} / A_m^{(n)} A_{m+1}^{(n)}$, $b_n = A_m^{(n)} A_{m+1}^{(n-1)}/A_m^{(n-1)} A_{m+1}^{(n)}$, $n \geq 1$,

and for $N \geq 2$ one obtains $D_N = \min_{1 \leq n \leq N} A_m^{(n)} A_{m+2}^{(n-1)}/A_{m+1}^{(n)} A_{m+1}^{(n-1)} > 0$, and

$Q_N = \max_{2 \leq n \leq N} A_{m+1}^{(n-2)} A_{m+1}^{(n)} / A_{m+1}^{(n-1)})^2 < 1$.

Especially for $f(z) = e^z$ Theorem 3 yields the

COROLLARY. Let $U_{m,n}(z)$ and $V_{m,n}(z)$ be the Padé-numerator and the Padé-denominator of $f(z) = e^z$. Then

U) For fixed $n \geq 0$ and $M \geq 2$ $U_{m,n}(z) \not\equiv 0$ for $1 \leq m \leq M$,

and all $z = r e^{i\vartheta}$ satisfying $\cos\vartheta \geq \frac{M-n-2}{M+n}$.

V) For fixed $m \geq 0$ and $N \geq 2$ $V_{m,n}(-z) \not\equiv 0$ for $1 \leq n \leq N$

and all $z = r e^{i\psi}$ satisfying $\cos\psi \geq \frac{N-m-2}{N+m}$.

This particular result was proved directly by E.B. Saff and R.S. Varga in [3] .

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