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EXACT ORDER OF BEST RATIONAL APPROXIMATION ON SOME CLASSES OF PERIODIC FUNCTIONS

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Let $W_{2\pi}^{r}V$ is the class of functions represented in the form

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_0^{2\pi} \mathcal{D}_r(x-t)h(t)\,dt, \quad r>0, \tag{1}$$

and $W_{2\pi}^{r}V_0$ is the class of functions represented by Stieltjes integral

$$f(x) = \frac{1}{\pi}\int_0^{2\pi} \mathcal{D}_{r+1}(x-t)\,dh(t), \tag{2}$$

where $h(t)$ is a function of bounded variation on the interval $[0,2\pi]$, $\left[\mathrm{Var}\; h(t)\right]_0^{2\pi} \le 1$, and

$$\mathcal{D}_r(t) = \sum_{k=1}^{\infty} k^{-r}\cos\left(kt - \frac{r\pi}{2}\right).$$

We denote by $\widetilde{W_{2\pi}^{r}V}$ and $\widetilde{W_{2\pi}^{r}V_0}$ the set of all functions represented in convolution form (1) or (2), where the kernel $\mathcal{D}_r(t)$ replaced by the conjugate kernel

$$\widetilde{\mathcal{D}}_r(t) = \sum_{k=1}^{\infty} k^{-r}\sin\left(kt - \frac{r\pi}{2}\right).$$

Let $R_n^T(f)$ is best rational approximation to the function $f(x)$, it is exactly

$$R_n^T(f) = \inf \|f(x) - z_n(x)\|,$$

where the lower bound is taken in the class of trigonometric rational functions with real coefficients of order not higher than n.

If there are positive constants C_1 and C_2 such that $C_1 d_n < \beta_n < C_2 d_n$, $n = \overline{1,\infty}$, then we write $d_n \asymp \beta_n$. Let K^r is any class of the set

$$W_{2\pi}^r V \ (W_{2\pi}^r V_0), \ \overline{W_{2\pi}^r V} \ (\overline{W_{2\pi}^r V_0}).$$

Theorem 1. $\sup\limits_{f \in K^r} R_n^T(f) \asymp \dfrac{1}{n^{r+1}}.$

As a tool for uniform approximation the author considers de la Vallee Poussin type operators [1] $V_{4n}(x,f)$, whose values are rational trigonometric functions of order at most $4n$. Let $\{w_k\}_{k=1}^n$, $|w_k| < 1$, be a system of complex numbers, $\arg w_k = \theta_k$, and

$$2\gamma_n(\theta) = n + \sum_{k=1}^{n} \frac{1 - |w_k|^2}{1 + |w_k|^2 - 2|w_k|\cos(\theta - \theta_k)}.$$

The rational operator $V_{4n}(x,f)$ is defined by the equalities

$$V_{4n}(x,f) = \frac{1}{4\pi\gamma_n(x)} \int_0^{2\pi} f(u)\, \Theta_{4n}(x,u)\, du,$$

$$\Theta_{4n}(x,u) = \left[\sin^2 \frac{u-x}{2}\right]^{-1} \cdot \left[\sin^2 2\int_x^u \gamma_n(\theta)\,d\theta - \sin^2 \int_x^u \gamma_n(\theta)\,d\theta\right]$$

Theorem 2. For any function $f(x) \in K^r$ there exists a system $\{w_k\}_{k=1}^n$ and a constant $C = C(r)$ such that

$$\|f(x) - V_{4n}(x,f)\| \leq \frac{C}{n^{r+1}}. \quad (3)$$

The convergence order in the estimate (3) is exact for any considering klass K^r since the following result is valid.

Theorem 3. There exist function $f_n^*(x) \in K^r$ such that

$$\| f_n^*(x) - V_{4n}(x, f_n^*) \| \geq C_0 n^{-(r+1)}, \quad C_0 > 0,$$

for any system $\{w_k\}_{k=1}^n$, $|w_k| < 1$.

Fourier type operator $S_{2n}(x,f)$ is defined by equality

$$S_{2n}(x,f) = \int_0^{2\pi} f(u) \cdot K_{2n}(x,u)\,du \quad \text{, wtere}$$

$$K_{2n}(x,u) = \left[2\pi \sin \frac{u-x}{2}\right]^{-1} \cdot \sin \int_x^u \left[2\gamma_n(\theta) + \frac{1}{2}\right] d\theta$$

Theorem 4. For any function $f(x) \in K^r$ there exists a system $\{w_k\}_{k=1}^n$ and a constant $C_1 = C_1(r)$ such that

$$\| f(x) - S_{2n}(x,f) \| \leq \frac{C_1 \ln n}{n^{r+1}}.$$

Rational approximation of periodical functions f having a fractional derivative of bounded variation (and conjugate to f) was investigated by the author in 1981-1983 years [2 - 4].

In nonperiodic case exact order of rational approximation on the classes V_r, $r=1,2,\dots$, was obtained [5] by V.A.Popov in 1976 year. For fractional r, $r > 0$, analogous results was obtained recently by A.P.Starovojtov.

References

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