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ISOMORPHISMS BETWEEN BESOV SPACES AND SEQUENCE SPACES

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The purpose of this paper is to describe isomorphisms between Besov spaces and sequence spaces. It was proved by J. Peetre and later by H. Triebel [3] that the Besov space on the unbounded domain in $\mathbb{R}^d$ is isomorphic to the space $l_q(l_p)$. Z.Ciesielski and T.Figiel [4] proved that the Besov space on the compact C^∞ manifold is isomorphic to the space $(\sum_{k=1}^{\infty} l_p^k)_q$.
The construction of isomorphisms described below is the same in both cases - the bounded or unbounded domain. The interpolating multidimensional splines are used in this construction.
Let us note that for the bounded domain the functions $(h_v)_{v\in\bigcup A_k}$ (see the proof of Theorem 4) form Schauder bases in Sobolev spaces and Besov spaces.
We shall consider the domains with minimally smooth boundary (for definition see [2]). Roughly speaking the boundary of $D\subset\mathbb{R}^d$ is minimally smooth, if it is locally isometric to the graph of the Lipschitz function $g:\mathbb{R}^{d-1}\longrightarrow\mathbb{R}$.
For the function f from the Sobolev space $W_p^n(D)$ we define the moduli of smoothness:

$$\omega_p^n(f,t)=\Big(t^{-d}\cdot\int_{B(0,t)}\int_{D_h}\sup_{|\alpha|=n}|\partial^\alpha f(x+h)-\partial^\alpha f(x)|^p\,dx\,dh\Big)^{1/p}$$

$$\overline{\omega}_p^n(f,t)=\sup_{h\in B(0,t)}\Big(\int_{D_h}\sup_{|\alpha|=n}|\partial^\alpha f(x+h)-\partial^\alpha f(x)|^p\,dx\Big)^{1/p}$$

where $D_h=\{x\in\mathbb{R}^d:[x,x+h]\subset D\}$, $B(0,t)=\{h\in\mathbb{R}^d:|h|<t\}$ and $\partial^\alpha f$ denotes the generalizated partial derivative of f.As usually for $p=\infty$ we change the integral into esssup.
Let $0<s<1$. The Besov space $B_{p,q}^{n+s}(D)$ is the space of all functions

$f \in W^n_p(D)$ such that

$$b^s_q(f) = \left(\int_0^1 \left(\bar{\omega}^n_p(f,t) \cdot t^{-s} \right)^q \frac{dt}{t} \right)^{1/q} < +\infty$$

with the usual change for $q = \infty$.

For $t > 0$ we shall denote by G_t the family of all cubes in $\mathbb{R}^d$ with vertices in the set $G^o_t = t \cdot \mathbb{Z}^d$ and with the length of edges equal to t.

H_n is the space of all polynomials g of d variables such that the degree of g as the polynomial of each variable separately is at most $2n+1$. The polynomial $g \in H_n$ is uniquely determined by its partial derivatives $\partial^\alpha g$ $(\max_i |\alpha_i| \leq n)$ at all vertices of some cube from G_t.

For $n \in \mathbb{N}$ and $e \in \{-1,1\}^d$ let $Z_{n,e} = \{(n_1,\dots,n_d) \in \mathbb{Z}^d : e_i \cdot n_i \geq 0; \sum_i |n_i| \leq n\}$

Let D be the domain in $\mathbb{R}^d$ with the minimally smooth boundary, $n \in \mathbb{N}$ and $t > 0$. Let $V_t = \{v \in G^o_t : \exists Q \in G_{2nt} \; v \in Q \subset D\}$. Let $D[t]$ be the sum of all cubes from G_t which intersect D. If t is sufficiently small, let us say $t \leq t(D)$, then $V_t \neq \emptyset$ and for each vertex $v \in (G^o_t \cap D[t]) \setminus V_t$ we can choose a vertex $r(v) \in V_t$ such that $|r(v) - v| \leq at$ (a depens on D and n only, we assume that $a \geq dn+1$). We put $r(v) = v$ for $v \in V_t$. For each $v \in V_t$, there exists $e \in \{-1,1\}^d$ such that $Z_v = v + tZ_{n,e} \subset V_t$.

Let $f \in C(D)$. For each vertex $v \in V_t$ let $g_v = g_v(f)$ be the unique polynomial of d variables of degree at most n such that $g_v | Z_v = f | Z_v$.

For each $Q \in G_t$ which is contained in $D[t]$ let $g_Q = g_Q(f)$ be the unique polynomial in H_n such that $\partial^\alpha g_Q(w) = \partial^\alpha g_{r(w)}(w)$ for all vertices w of the cube Q and all multiindices α such that $\max_i |\alpha_i| \leq n$.

The derivatives of g_{Q_1} and g_{Q_2} coincides on the common face of Q_1 and Q_2. Hence the function $L_t f : D \longrightarrow \mathbb{R}$ given by the formula

$$L_t g(x) = g_Q(x) \quad \text{for } Q \subset D[t], \quad Q \in G_t \; ; x \in D \cap Q$$

is well defined and moreover belongs to $C^n(D)$. The above construction in the very particular case (D equal to a cube), was described by W.S. Rjabenkij [1]. The properties of the operator L_t are summarized below:

<u>Proposition 1</u>:

(a) L_t is the linear operator from $C(D)$ into $C^n(D)$.

(b) $f | V_t = 0 \Rightarrow L_t f = 0$

(c) $L_t f | V_t = f | V_t$

(d) $f(x) = g(x)$ for $|x-y| \leq 2at \Rightarrow L_t f(x) = L_t g(x)$

(e) For $p > \frac{d}{n}$ there exists $c_1 > 0$ such that for $0 < t \leq t(D)$, $f \in W^n_p(D)$ and $s > 0$ we have

$$\omega^n_p(L_t f, s) \leq c_1 s t^{-1} \omega^n_p(f,t) \tag{1}$$

(2) $$\|L_t f\|_{W^m_p(D)} \le c_1 \|f\|_{W^m_p(D)}$$

The set V_t has the additional property

Proposition 2: Let $p > \frac{d}{n}$. There exists $c_2 > 0$ such that for $0 < t \le t(D)$ $|\alpha| \le n$ and $f \in W^n_p(D)$ such that $f|V_t = 0$ we have

(3) $$\Big(\int_D |\partial^\alpha f(x)|^p\,dx\Big)^{1/p} \le c_2 t^{n-|\alpha|} \cdot \omega^n_p(f,t)$$

(4) $$\sup_{x\in D} |f(x)| \le c_2 t^n \Big(\int_D \sup_{|\alpha|=n} |\partial^\alpha f(x)|^p\,dx\Big)^{1/p}$$

Let $r \in \mathbb{N}$, $r \ge t(D)^{-1}$. For $k = 1,2,\dots$ we denote

$$U_k = V_{r^{-k}} \ (U_0 = \emptyset),\quad A_k = U_k \setminus U_{k-1},\quad L_k = L_{r^{-k}}$$

Let $f \in W^n_p(D)$. We denote: $R_0 f = f$; $S_k f = {}_k R_{k-1} f$, $R_k f = R_{k-1} f - S_k f$ for $k = 1,2,\dots$. Let $E_{k,p} = S_k(W^n_p(D))$. Of course $E_{k,p} = \{f \in L_k(W^n_p(D)) : f|U_{k-1} = 0\}$. For $0 < s < 1$, $1 \le q \le \infty$ let

$$\tilde{b}^s_q(f) = \Big(\sum_{k=1}^{\infty} \big(\|S_k f\|_{W^m_p(D)} \cdot r^{sk}\big)^q\Big)^{1/q}$$

Theorem 3: The space $E_{k,p}$ is isomorphic to $l_p(A_k)$. More precisely: let $\frac{d}{n} < p < \infty$, there exists $c_3 > 0$ such that for $k = 1,2,\dots$ and $g \in E_{k,p}$

(5) $$c_3^{-1} r^{k(\frac{d}{p}-n)} \Big(\sum_{v\in A_k} |g(v)|^p\Big)^{1/p} \le \|g\|_{W^m_p} \le c_3 r^{k(\frac{d}{p}-n)} \Big(\sum_{v\in A} |g(v)|^p\Big)^{1/p}$$

Theorem 4: Let $r > (c_1+1)^{\frac{1}{1-s}}$, $\frac{d}{n} < p < \infty$. The norm $\tilde{b}^s_q + \|\cdot\|_{W^m_p}$ is equivalent to the norm $\tilde{b}^s_q + \|\cdot\|_W$ of the Besov space B^{n+s}_{pq}.

Theorem 5: Let $\frac{d}{n} < p < \infty$, $1 \le q \le \infty$, $0 < s < 1$. The space B^{n+s}_{pq} is isomorphic either to $l_q(l_p)$ when the domain D is unbounded or to $\big(\sum_{n=1}^{\infty} l^n_p\big)_q$, when the domain D is bounded.

Theorem 5 is an easy consequence of Theorems 3 and 4. If D is bounded then the sets A_k are finite. Hence B^{n+s}_{pq} is isomorphic to $\big(\sum_k l_p^{\operatorname{card} A_k}\big)_q$. If D is unbounded then the sets A_k are infinite and the spaces E_{kp} are isomorphic to l_p.

Let us sketch the proofs of Theorems 3 and 4.

Proof of Th. 3: For $v \in A_k$ let h_v be the unique function in E_{kp} such that $h_v(v) = 1$, $h_v(w) = 0$ for $w \in A_k \setminus \{v\}$. In fact the function h_v does not depend on p. From Proposition 1 and (4) it follows that

(6) $$\operatorname{diam}(\operatorname{supp} h_v) \le 4ar^{-k}\ ; \quad \sup_{x\in D} |\partial^\alpha h_v(x)| \le c_4 r^{k|\alpha|} \quad \text{for } |\alpha| \le n+1$$

The function $g \in E_{kp}$ is determined by its values on A_k so $g = \sum_{v\in A} g(v)\cdot h_v$. Of course it is enough to prove (5) for $g \in E_{kp}$ such that the set $\{v \in A_k : g(v) \ne 0\}$ is finite. Let us fix the function g.

Let $A_Q = \{v \in A_k : \operatorname{supp} h_v \cap Q \neq \emptyset\}$ for $Q \in F_k = G_{r^{-k}} \cap \{Q : Q \cap D \neq \emptyset\}$

We have for $|\alpha| \leq n$

$$\int_D |\partial^\alpha g(x)|^p\, dx \leq \sum_{Q \in F_k} \int_Q \Big|\sum_{v \in A_Q} g(v)\, \partial^\alpha h_v(x)\Big|^p dx \leq c \sum_{v \in A_k} |g(v)|^p\, r^{k(pn-d)}$$

The last inequality follows from (6) and the fact that card $A_Q \leq (4a)^d$. The right hand side of (5) follows from the above inequality.

Let $A_w = \{v \in A_k : \operatorname{supp} h_v \cap \operatorname{supp} h_w \neq \emptyset\}$ for $w \in A_k$. Hence card $A_w \leq$ $\leq N = (8a)^d$. Let $A^o = \{w \in A_k : \sup_{v \in A_w} |g(v)| \leq 2N|g(w)|\}$

Let $T : A_k \setminus A^o \to A_k$ be such that $g(T(w)) \geq 2N|g(w)|$ and $T(w) \in A_w$ for $w \in A_k \setminus A^o$. For $w \in T^{j+1}(A_k \setminus A^o)$ we have

$$\sum_{v \in T^{-(j+1)}(w)} |g(v)|^p \leq \sum_{u \in T^{-j}(w)} \sum_{v \in T^{-1}(u)} |g(v)|^p \leq 2^{-p} \sum_{u \in T^{-j}(w)} |g(u)|^p \leq \ldots \leq$$

$$\leq 2^{-p(j+1)} |g(w)|^p$$

But $A_k = \bigcup_{j=0}^{\infty} T^{-j}(A^o)$. From the above inequality we obtain:

$$(7) \qquad \sum_{v \in A_k} |g(v)|^p \leq \sum_{j=0}^{\infty} \sum_{v \in T^{-j}(A^o)} |g(v)|^p \leq \Big(\sum_{v \in A^o} |g(v)|^p\Big) \cdot 2^p$$

It follows from (1) that $|h_v(x) - h_v(y)| \leq c_5 |x-y| \cdot r^k$ for $v \in A_k$. Hence for $v, w \in A_k$ and $|x-w| \leq r^{-k} \cdot (4 \cdot c_5 \cdot N^2)^{-1}$, we have $|h_v(x) - h_v(w)| \leq$ $\leq (4N^2)^{-1}$. In particular for $w \in A^o$ and $|x-w| \leq r^{-k} \cdot (4c_5 N^2)^{-1}$ it holds

$$|g(x)| \geq |g(w)|\, |h_w(x)| - \sum_{v \in A_k \setminus \{w\}} g(v)\, |h_v(x)|$$

$$\geq \tfrac{3}{4} g(w) - N \cdot 2N \cdot |g(w)| \tfrac{1}{4N^2} \geq \tfrac{1}{4} |g(w)|$$

Hence

$$(8) \qquad \Big(\sum_{w \in A^o} |g(w)|^p\Big)^{1/p} \leq c r^{k\frac{d}{p}} \|g\|_{L_p(D)} \cdot c r^{k(\frac{d}{p}-n)} \|g\|_{W_p^n(D)}$$

The last inequality is the consequence of (3) $g|U_{k-1} = 0$, for $k = 1$ we should change c. The left hand side of (5) follows from (7) and (8).

<u>Proof of Th. 4</u>

<u>Lemma 6</u> Let $|b| < 1$. The operator P_b given by the formula

$$P_b((t_k)) = \Big(\sum_{j=1}^{\infty} t_j\, b^{|j-k|}\Big)$$

is bounded as an operator from l_q into l_q $(1 \leq q \leq \infty)$ and $\|P_b\| \leq \frac{2}{1-|b|}$

Let $f \in B^{n+s}_{pq}(D)$, $1 \leq q < \infty$. We have

$$r^{s(k+1)}\|S_{k+1}f\|_{W_p^n(Q_{k+1})} \leq c\cdot r^{s(k+1)}\omega_p^n(S_{k+1}f, r^{-k}) \leq cr^{s(k+1)}\omega_p^n(R_kf, r^{-k-1}) \leq$$

$$\leq cr^{s(k+1)} r_1^{-k} \cdot \sum_{j=1}^{k+1} r_1^{\,j}\,\omega_p^n(f, r^{-j}) \leq cr^s \sum_{j=1}^{k+1} r_2^{\,k-j}\,\overline{\omega}_p^n(f, r^{-j}) r^{sj} \leq$$

$$\leq c\sum_{j=1}^{k+1} r_2^{\,k-j}\Big(\int_{r^{-j}}^{r^{-j+1}} (\overline{\omega}_p^n(f,u)\,u^{-s})^q \frac{du}{u}\Big)^{1/q}$$

where $r_1 = r(c_1+1)$, $r_2 = r^s \cdot r_1$

The first inequality follows from (3) , the second one from (1), the third one as the corollary from (1). From Lemma 6 (for $b = r_2$) and the above inequality we obtain:

$$\tilde{b}_q^s(f) \leq c\; b_q^s(f)$$

Now let $f \in W_p^n(D)$ be such that $\tilde{b}_q^s(f) < \infty$, $|\alpha| = n$, $h \in \mathbb{R}^d$, $x \in D_h$. For $k > 1$ we have

$$|\partial^\alpha S_k f(x+h) - \partial^\alpha S_k f(x)| \leq c \max_{v \in A_k} |S_k f(v)| \cdot \max_{v \in A_k} |\partial^\alpha h_v(x+h) - \partial^\alpha h_v(x)| \leq$$

$$\leq c \sup_{x \in D} |S_k f(x)|\,|h| \cdot r^{k(n+1)} \leq c|h| \cdot \|S_k f\|_{W_p^n(D)} \cdot r^k$$

The first two inequalities follow from (6), the last one from (4). For $k = 1$ we can obtain the same changing c.

$$\Big(\int_D \sup_{|\alpha|=n} |\partial^\alpha S_k f(x+h) - \partial^\alpha S_k f(x)|^p\, dx\Big)^{1/p} \leq c|h| \cdot r^k \|S_k f\|_{W_p^n}$$

The left hand side of the last inequality is bounded by

$2 \cdot \|S_k f\|_{W_p^n(D)}$ too.

Hence for $t \leq r^{-k}$ we have

$$\overline{\omega}_p^n(f,t) \leq \sup_{|h| \leq t} \sum_{j=1}^{\infty} \Big(\int_{D_h} \sup_{|\alpha|=n} |\partial^\alpha S_j f(x+h) - \partial^\alpha S_j f(x)|^p\, dx\Big)^{1/p} \leq$$

$$\leq c \cdot \sum_{j=1}^{k} t\, r^j \|S_j f\|_{W_p^n(D)} + \sum_{j=k+1}^{\infty} \|S_j f\|_{W_p^n(D)}$$

and

$$\Big(\int_{r^{-k-1}}^{r^{-k}} (\overline{\omega}_p^n(f,t)\, t^{-s})^q \frac{dt}{t}\Big) \leq c\Big(\sum_{j=1}^{k} \|S_j f\|_{W_p^n} \cdot r^{sj}(r^{s-1})^{k-j} +$$

$$+ \sum_{j=k+1}^{\infty} \|S_j f\|\; r^{sj}\,(r^{-s})^{j-k}\Big)$$

Therefer from Lemma 6 $\left(\text{for } b = \max\ r^{-s}, r^{s-1}\right)$ we obtain

$$\Big(\int_0^{r^{-1}} (\tilde{\omega}_p^n(f,t)\, t^{-s})^q \frac{dt}{t}\Big)^{1/q} \leq c\, \tilde{b}_q^s(f)..$$

But

$$\Big(\int_{r^{-1}}^{1} (\tilde{\omega}_p^n(f,t) \cdot t^{-s})^q \frac{dt}{t}\Big)^{1/q} \leq c\, \|f\|_{W_p^n}$$

Hence $\quad b_q^s(f) \leq c \cdot \tilde{b}_q^s\, f$

For $q = \infty$ the proof is almost the same.

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