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ON THE HAHN POLYNOMIALS APPLICATION FOR THE OPTIMAL TABULATION OF FUNCTIONS

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1. The classical orthogonal Hahn polynomials of discrete variable may be defined [1] by the following equality ($\alpha, \beta > -1$, $0 \le n \le N-1$)

$$Q_n(x) = Q_n(x; \alpha, \beta, N) = \sum_{k=0}^{n} \frac{(-n)_k (-x)_k (n+\alpha+\beta+1)_k}{(\alpha+1)_k (-N+1)_k \, k!},$$

where $(a)_0 = 1$, $(a)_k = a \cdots (a+k-1)$.

Let

$$\rho(x) = \rho(x; \alpha, \beta, N) = \frac{\Gamma(N)\Gamma(\alpha+\beta+2)\Gamma(x+\alpha+1)\Gamma(N-x+\beta)}{\Gamma(N+\alpha+\beta+1)\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(x+1)\Gamma(N-x)},$$

$$\pi_n = \pi_n(\alpha, \beta, N) = (2n+\alpha+\beta+1) \times$$

$$\frac{\Gamma(\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha+\beta+2)\Gamma(\alpha+1)\Gamma(n+\beta+1)\, n!} \cdot \frac{\binom{N-1}{n}}{\binom{N+\alpha+\beta+n}{n}}.$$

Then

$$\sum_{i=0}^{N-1} Q_n(i) Q_m(i) \rho(i) = \begin{cases} 0 & (n \ne m), \\ \frac{1}{\pi_n} & (n = m). \end{cases}$$

The Hahn polynomials are the discrete analogue of the classical Jacoby polynomials $P_n^{(\beta,\alpha)}(x)$, in particular, $Q_n(x; -\frac{1}{2}, -\frac{1}{2}, N)$ is the discrete analogue of the Čebyshev polynomials $T_n(x) = \cos(n \arccos x)$

2. Let $A_\rho(M)$ - be the space of functions which are analytical

inside the ellipse E_ρ with the foci at the points $x = \pm 1$ and the half-sum of the axes, equal to $\rho > 1$, real in the large axis and bounded inside E_ρ by the constant M. Let us consider the problem of the optimal tabulation of functions $f \in A_\rho(M)$.

Following [2], denote by (T_x, R) the table of the element $x \in X \subset B$, where X is the compactum of the normed space B. Let D_n be the set of binary words with the lengths not exceeding n, $T = T(x) = T_x$ is the reresentation of the compactum X in the set D_n. The pair (T, R) is called the tabulation method of the elements from X, while least upper bound of the binary words from the image $T(X)$ is its volume. The quantity

$$\mathcal{E}_X(T, R) = \sup_{x \in X} \| x - R(T_x) \|$$

is called the accuracy of the method (T, R). Denote by $H_\varepsilon(X)$ the absolute ε-entropy of the compactum X. Let $(T^\varepsilon, R^\varepsilon)$ be the tabulation method of the elements from X with $\mathcal{E}_X(T^\varepsilon, R^\varepsilon) \le \varepsilon$, d^ε being its volume. A.Vitushkin [2] showed that

$$d^\varepsilon \geqslant H_\varepsilon(X) \tag{1}$$

and there exists a method $(T^\varepsilon, R^\varepsilon)$, for which

$$d^\varepsilon = [H_\varepsilon(X)] + 1, \tag{2}$$

where $[\gamma]$ is the integer part of γ.

The correlations (1) and (2) naturally result in the following

D e f i n i t i o n. The system of methods $(T^\varepsilon, R^\varepsilon)$ $(\varepsilon \to 0)$ is called asymtoticlly optimal on the compactum X if the following equality holds

$$\lim_{\varepsilon \to 0} \frac{d^\varepsilon}{[H_\varepsilon(X)] + 1} = 1.$$

As it follows from the results obtained by A.Vitushkin the asymptotically optimal method for tabulation of functions $f \in A_\rho(M)$ is the "memorisation" of Fourier-Čebyshev coefficients

$$a_k(f) = \frac{2}{\pi} \int_{-1}^{1} \frac{f(t) T_k(t)}{\sqrt{1 - t^2}} dt. \tag{3}$$

On the other hand the calculation of the coefficients $a_K(f)$ bythe formula (3) faces some considerable difficulty in practice. In the present paper we study the asymptotically optimal method for tabulation of functions $f \in A_\rho(M)$, which consists in the "memorisation" of the Fourier-Hahn coefficients

$$a_{K,N}(f) = \pi_K \sum_{K=0}^{N-1} \rho(i) f\left(-1+\frac{2i}{N-1}\right) Q_K(i) \qquad (4)$$

where $\rho(x) = \rho(x; -\frac{1}{2}, -\frac{1}{2}, N)$, $\pi_K = \pi_K(-\frac{1}{2}, -\frac{1}{2}, N)$, $Q_K(x) = Q_K(x; -\frac{1}{2}, -\frac{1}{2}, N)$.

Calculation of the coefficients $a_{K,N}(f)$ using the formula (4) couses no difficulty and is particularly convenient if the recurrent rellations for the polynomials $Q_n(x)$ [1] are taken into account.

3. Vitushkin's method is essentially based on the boundness in $C[-1,1]$ of the Čebyshev polynomials system $\{T_n\}_0^\infty$. The Hahn polynomials system

$$Q_n\left(\frac{N-1}{2}(1+t); -\frac{1}{2}, -\frac{1}{2}, N\right) \quad (0 \le n \le N-1,\ N=2,3,\dots)$$

is not bounded in $C[-1,1]$. Moreover it is shown in [3] that quantity

$$Q(n,N,\alpha,\beta) = \max_{-1\le x\le 1} \left| Q_n\left(\frac{N-1}{2}(1+x); \alpha, \beta, N\right)\right| \qquad (5)$$

increases with $n^2/N \to \infty$ at a speed which exceeds $C\exp(\sigma n^2/N)$ $(\sigma > 0)$, where C , $C_K(\alpha, \beta, \dots, M)$ are positive constants depending only on the specifed parameters.

On the other hand, in [4] , in particular, the following estimate was obtained $(-1 \le x \le 1)$:

$$\binom{n+\alpha}{n} \left| Q_n\left(\frac{N-1}{2}(1+x); \alpha, \beta, N\right)\right| (1+x)^{\alpha/2+1/4} (1-x)^{\beta/2+1/4} \le \qquad (6)$$

$$\le \frac{C(a,\alpha,\beta)}{\sqrt{n}} \quad \left(\alpha, \beta \ge -\frac{1}{2},\ a>0,\ 1 \le n \le a N^{1/3}, N=2,3,\dots\right).$$

Introduce the notations

$$Q(n,N) = Q\left(n,N,-\frac{1}{2},-\frac{1}{2}\right),\quad P(n,N) = \max_{0\le K\le n} Q(K,N), \qquad (7)$$

$$q(n,N)=\max_{0\le x\le N-1}\left|Q_n(x-1;\tfrac{1}{2},-\tfrac{1}{2},N-1)\right|. \qquad (8)$$

From (5)-(8) we deduce

$$1\le p(n,N)\le c_1(a),\ n\,q(n,N)\le c_2(a)$$
$$(0\le n\le aN^{1/3},\ N=2,3,\dots).$$

4. Proceed now to the construction of the asymptotically optimal method for tabulation of the class $A_\rho(M)$ functions based on the Hahn polynomials application. The idea of the method consist in memorising the approximate values of the Fourier-Hahn coefficients.

Let $\delta>0$, $a>0$

$$\gamma_1=\gamma_1(a,M,\rho)=\frac{4M\,c_1(a)\,\rho}{\rho-1}.$$

Now choose the natural numbers n, N, μ_K, $K=0,\dots,n$ from the condition

$$0\le n\le a\,N^{1/3},\ \delta(\mu_K-1)\le\gamma_1\rho^{-K}\le\delta\mu_K\ (0\le K\le n).$$

Next, let the integers $\tau_0,\dots,\tau_n$ be such so as the following is true $|\tau_K|\le\mu_K$ $(K=0,1,\dots,n)$.

Assuming

$$\Psi_{n,N}=\Psi_{n,N}(x;\delta,\tau_0,\dots,\tau_n)=\sum_{K=0}^{n}\delta\,\tau_K\,Q_{K,N}(x)$$

and $f\in A_\rho(M)$ we find the integers $\tau_0,\dots,\tau_n$ which satisfy the condition

$$|a_{K,N}(f)-\tau_K\delta|\le\tfrac{1}{2}\delta. \qquad (9)$$

Denote by T_f the binary word resulting from substituting for the component of the vector $(\tau_0,\dots,\tau_n)$ their binary forms, where the first binary symbols denote the signs of the corresponding numbers τ_K $(K=0,\dots,n)$. Copare each element $f\in A_\rho(M)$ with the table $(T_f,\ \Psi_{n,N}(\delta,f))$, where $\Psi_{n,N}(\delta,f)=\Psi_{n,N}(x;\delta,\tau_0,\dots,\tau_n)$ and the numbers $\tau_0,\dots,\tau_n$ satisfy the inequality (9).

Let

$$\gamma_2=2M\,(c_1(2a))^2\cdot\sup_{0\le n<\infty}\left(\frac{2\rho}{(\rho-1)^2}+\frac{4n+3}{\rho^n(\rho-1)}\right).$$

THEOREM 1. Let $0<\varepsilon\le 1,\ a>0,$

$$n=\left[\log_{\varrho}\frac{2(\gamma_2+1)}{\varepsilon}+1\right]+1<aN^{1/3}\le\frac{N-1}{2},\quad \delta=\frac{\varepsilon}{(n+1)C_1(a)},$$

$$T^{\varepsilon}=T^{\varepsilon}(f)=T_f,\quad S^{\varepsilon}_{n,N}=S^{\varepsilon}_{n,N}(T_f)=\Psi_{n,N}(\delta(\varepsilon),f).$$

Then $(T^{\varepsilon},S^{\varepsilon}_{n,N})$ $(\varepsilon\to 0)$ is an asymptotical optimal system of methods for tabulation of the class $A_{\varrho}(M)$ functions.

5. Let the values $v_i=f(x_i)+h_i$ $(i=0,\dots,N-1)$ be given instead of the exact values $f(x_i)$ $(i=0,\dots,N-1)$. Then it is possible, by anology with the above - described procedure, to construct the asymptotically optimal method for tabulation of the class $A_{\varrho}(M)$ functions based on memorising the coefficients

$$\tilde{a}_{K,N}(f)=\pi_K\sum_{i=0}^{N-1}\varrho(i)\,v_i\,Q(i).$$

Let

$$S_{n,N}(f,x)=\sum_{K=0}^{n}a_{K,N}(f)\,Q_K\left(\frac{N-1}{2}(1+x)\right),$$

$$L_{n,N}(x)=\sup\left\{\|S_{n,N}(f)\|_{C[-1,1]}:f\in C[-1,1]\right\}$$

The proof of the assumption mentioned is based on the following result.

THEOREM 2. Let $0\le t\le\frac{N-1}{2},\ 2\le n\le N-2$. Then

$$L_{n,N}\left(\frac{2t}{N-1}-1\right)\le\frac{2(N-n)(2n+1)}{\sqrt{N-1}}\,q(n,N)\,Q(n+1,N)\,B_N\times$$

$$\left[\frac{4\ln n}{\sqrt{N-t}}+\frac{n\,(n^2+N-1)^{1/2}}{(N-1)\,(N+(N-1)/n^2-t)^{1/2}}+\frac{n}{N-1}+\right.$$

$$\left.+\frac{1}{\sqrt{N-t}}\cdot\ln\frac{4(N-t)}{N-1}+\frac{n^2}{(N-1)(N-t-(N-1)/n^2)^{1/2}}+\frac{1}{N-1-t}\right]+$$

$$+2B_N\,(p(n,N))^2\,(n+1)\left[\frac{\sqrt{2}\,\pi}{n}+\frac{2}{(N-t-(N-1)/n^2)^{1/2}}\right],$$

where $B_N=\max\limits_{0\le i\le N-1}\varrho(i)\sqrt{(i+1)(N-i)}$.

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