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APPROXIMATION IN THE HARDY SPACE $\mathscr{H}^1(D)$ FROM LOCAL BOUNDARY VALUES

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1. Introduction. It is a pervasive and fundamental fact that the values of a function f which is holomorphic in a non-void open subset D of the complex plane $\mathbb{C}$ cannot vanish on a large subset of D unless the function f is null. A first elementary version of this well-known statement is the simple result that the zeros of the holomorphic function $f \neq 0$ cannot accumulate within D. In the present paper we are interested in the boundary values $\tilde{f}$ of functions f that are holomorphic in the open unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$ of $\mathbb{C}$. Subject to certain hypotheses on the functions f, the boundary values $\tilde{f}$ of f on $\partial D = \mathbb{T}$ cannot be zero on a set Ω of positive Lebesgue measure $\lambda(\Omega)$ unless f=0. In different degrees of generality and precision, this result is due to Fatou, Ostrowski, F. and M. Riesz, and Szegö. More precisely, if $(\mathscr{H}^p(D))_{p \in [1,\infty]}$ denotes the scale of Hardy spaces, local boundary values on $\mathbb{T}$ of functions $f \in \mathscr{H}^p(D)$, $p \in [1,\infty]$, determine f uniquely (cf. Theorem 1 and the Corollary of Theorem 6 infra). Indeed, in the case $p \in]1,\infty[$ a procedure due to Patil allows to construct explicitly approximations of the functions $f \in \mathscr{H}^p(D)$ with respect to the Hardy norm $\|.\|_{\mathscr{H}^p(D)}$ from local boundary values on $\mathbb{T}$ (see Section 3 infra). In the case p=1, however, the main difficulty in constructing Patil type approximations of functions $f \in \mathscr{H}^1(D)$ with respect to $\|.\|_{\mathscr{H}^1(D)}$ from local boundary values on $\mathbb{T}$ comes from the fact that by virtue of Newman's theorem there does not exist any continuous projector of $L^1(\mathbb{T})$ onto $\mathscr{H}^1(\mathbb{T})$. Thus norm continuous Toeplitz operators on $\mathscr{H}^1(\mathbb{T})$ with symbols merely belonging to $L^\infty(\mathbb{T})$ are not available. The basic tool to overcome the difficulty mentioned above will be Toeplitz operators $\mathscr{H}^1(\mathbb{T}) \to \mathscr{H}^1(\mathbb{T})$ with symbols that are smooth enough to multi-

ply the (strong) dual BMO($\mathbb{T}$) (= vector space of all complex-valued functions on $\mathbb{T}$ with bounded mean oscillation) of the Hardy subspace $\mathscr{H}^1(\mathbb{T})$ of $L^1(\mathbb{T})$.

2. Hardy spaces. Denote by $\mathscr{H}(D)$ the complex vector space of all functions that are holomorphic in D. For $f \in \mathscr{H}(D)$ and $r \in [0,1[$ let the assignment

$$f_r : \mathbb{T} \ni w \rightsquigarrow f(rw) \in \mathbb{C}$$

be the "dilation" of the function f - then, for any exponent $p \in [1,\infty[$, set

$$\mathscr{H}^p(D) = \{f \in \mathscr{H}(D) \mid \|f\|_{\mathscr{H}^p(D)} < +\infty\}$$

where the Hardy norm $\|.\|_{\mathscr{H}^p(D)}$ is defined via the prescription

$$\begin{aligned} \|f\|_{\mathscr{H}^p(D)} &= \sup_{r \in [0,1[} \|f_r\|_p \\ &= \sup_{r \in [0,1[} \left(\int_{\mathbb{T}} |f(rw)|^p d\lambda(w)\right)^{1/p} \end{aligned}$$

and λ denotes the Haar measure of $\mathbb{T}$ normalized by $\lambda(\mathbb{T})=1$. Set in addition

$$\mathscr{H}^\infty(D) = \{f \in \mathscr{H}(D) \mid \|f\|_\infty < +\infty\}$$

where as usual

$$\|f\|_\infty = \sup_{z \in D} |f(z)|.$$

Then the family $(\mathscr{H}^p(D))_{p \in [1,\infty]}$ of complex Banach spaces is called to be the scale of Hardy spaces on the open unit disc D. Of course, the canonical injections

$$\mathscr{H}^q(D) \hookrightarrow \mathscr{H}^p(D) \qquad (1 \le p \le q \le \infty)$$

are continuous. Moreover, for every function $f \in \mathscr{H}^p(D)$ the pointwise radial limits of the dilations

$$\tilde{f}(w) = \lim_{r\to 1-} f_r(w)$$

exists for λ-almost all points $w \in \mathbb{T}$ and the $\mathbb{C}$-linear mapping defined via the prescription

$$\mathcal{H}^p(D) \ni f \longmapsto \tilde{f} \in L^p(\mathbb{T})$$

is an isometry for every exponent $p \in [1,\infty]$. Its image $\mathcal{H}^p(\mathbb{T})$ is a closed vector subspace of $L^p(\mathbb{T})$ and will be called the Hardy subspace of $L^p(\mathbb{T})$. If we denote for every function $g \in L^1(\mathbb{T})$ by $(\hat{g}(n))_{n \in \mathbb{Z}}$ the bi-infinite sequence of its Fourier coefficients, then the equivalent characterization

$$\mathcal{H}^p(\mathbb{T}) = \{\tilde{f} \in L^p(\mathbb{T}) \mid \hat{\tilde{f}}(n) = 0 \text{ for } n<0\}$$

holds for all exponents $p \in [1,\infty[$.

Theorem 1 (Fatou, Ostrowski, F. and M. Riesz, Szegö). Let Ω be a subset of $\mathbb{T}$ such that $\lambda(\Omega) > 0$. If $f \in \mathcal{H}^p(D)$ and $g \in \mathcal{H}^p(D)$, where $p \in [1,\infty]$, are functions such that their radial limits coincide on Ω, i.e., such that the identity

$$\tilde{f}(w) = \tilde{g}(w)$$

holds for all points $w \in \Omega$ - then $f = g$.

One procedure to prove this theorem is to combine Jensen's inequality and Fatou's lemma in order to establish that $h \in \mathcal{H}^p(D)$, $h \neq 0$ implies $\log|\tilde{h}| \in L^1(\mathbb{T})$; see, for instance, Helson [1] or Katznelson [2]. In the Hilbert space case (n=2), however, the following elegant approach due to Helson is more in the vein of approximation theory. If $\tilde{h} \in \mathcal{H}^2(\mathbb{T})$ is different from the null function we may suppose $\hat{\tilde{h}}(0) = 1$ without restriction of generality. Let S denote the closure of the subset $\{\tilde{p}_n\tilde{h} \mid n\geq 1\}$ of $\mathcal{H}^2(\mathbb{T})$ where $\tilde{p}_n$ denotes a trigonometric polynomial of the form

$$\tilde{p}_n(w) = 1 + \sum_{1\leq j\leq n} \tilde{c}_j w^j \qquad (w \in \mathbb{T})$$

with complex coefficients $(\tilde{c}_j)_{1\leq j\leq n}$. Since S is a non-void closed convex subset of the complex Hilbert space $L^2(\mathbb{T})$ it admits an element $\tilde{h}_o \in S$ of <u>smallest</u> norm. For any number $\tilde{c} \in \mathbb{C}$ and any integer $n \geq 1$ the function $w \rightsquigarrow (1+\tilde{c}w^n)\tilde{h}_o(w)$ belongs to S. Hence the quadratic expression

$$\int_{\mathbb{T}} |(1+\tilde{c}w^n)\tilde{h}_o(w)|^2 d\lambda(w) = \int_{\mathbb{T}} |\tilde{h}_o(w)|^2 d\lambda(w) + 2\mathrm{Re}\left(\tilde{c}\int_{\mathbb{T}} |\tilde{h}_o(w)|^2 w^n d\lambda(w)\right) + \tilde{c}^2 \int_{\mathbb{T}} |\tilde{h}_o(w)|^2 d\lambda(w)$$

in the variable $\tilde{c}$ has a minimum at $\tilde{c}=0$. Thus we obtain the orthogonality

$$\int_{\mathbb{T}} |\tilde{h}_o(w)|^2 w^n d\lambda(w) = 0$$

for all integers $n \geq 1$. By taking the complex conjugate we see that all the Fourier coefficients of index $n \neq 0$ of $|\tilde{h}_o|^2 \in L^1(\mathbb{T})$ vanish. Thus $|\tilde{h}_o|^2$ is a constant ≥ 0 on $\mathbb{T}$. If $\tilde{h}$ vanishes on Ω we conclude that $\tilde{h}_o = 0$. On the other hand we have $\tilde{h}_o \in S$ and therefore $\int_{\mathbb{T}} \tilde{h}_o(w) d\lambda(w) = 1$. This contradiction proves the theorem in the case n=2, i.e., for the complex Hilbert space

$$\mathscr{H}^2(D) = \{f \in \mathcal{H}(D) \mid f(z) = \sum_{n \geq 0} a_n z^n, (a_n)_{n \in \mathbb{Z}} \in \ell^2_{\mathbb{C}}\}$$

where $z = rw \in D$ and $r \in [0,1[$.

<u>3. Recovering from local boundary values.</u> Theorem 1 supra makes it apparent that the functions $f \in \mathcal{H}^p(D)$, $p \in [1,\infty]$, are determined uniquely by their restrictions $\tilde{f}|\Omega$ provided $\lambda(\Omega) > 0$. Since the Poisson kernel $(\Pi_r)_{r \in [0,1[}$ provides the harmonic (which is actually the holomorphic) extension $f = \Pi_r * \tilde{f}$ of $\tilde{f} \in \mathcal{H}^p(\mathbb{T})$ to the interior D of the unit circle ∂D, the following recovering problem arises: Is there an analogue of the Poisson integral for reconstructing $f \in \mathcal{H}^p(D)$ from its local boundary values $\tilde{f}|\Omega$? The question was answered in the affirmative by Patil [4] who studied an algorithm for recovering all of f from $\tilde{f}|\Omega$. More precisely, Patil constructed explicitly a family of functions $(g_\lambda)_{\lambda > 0}$ in $\mathcal{H}^p(D)$, $p \in]1,\infty[$, depending only upon Ω and the restriction $\tilde{f}|\Omega$ such that

$$\lim_{\lambda\to\infty} \| f-g_\lambda \|_{\mathcal{H}^p(D)} = 0$$

holds for all exponents $p \in]1,\infty[$. Patil's proof (loc. cit.) uses basically the theory of Toeplitz operators and the M. Riesz projection theorem. In the case p=1, however, the M. Riesz theorem fails. Indeed, Newman [3] proved by using the averaging principle (cf. [6]) that there does not exist any continuous projector of $L^1(\mathbb{T})$ onto $\mathcal{H}^1(\mathbb{T})$. Thus Patil's construction does not give a suitable norm approximation of $f \in \mathcal{H}^1(D)$.

The main idea to overcome the difficulties in the case p=1 is to replace the M. Riesz projection theorem by the Fefferman-Stein duality

$$\mathcal{H}^{1'}(\mathbb{T}) = \mathrm{BMO}(\mathbb{T}).$$

If φ denotes a complex-valued function on $\mathbb{T}$ that is smooth enough - then φ is a pointwise multiplier of BMO($\mathbb{T}$) and the Toeplitz operator P_φ with symbol φ is continuous on the Hardy space $\mathcal{H}^1(\mathbb{T})$.

<u>4. Toeplitz operators with periodic $\mathcal{C}^1$-symbols.</u> Let Ω denote a non-void open subset of $\mathbb{T}$ and $\rho \in \mathcal{C}^2(\mathbb{T})$ a function such that

$$\mathrm{Supp}(\rho) \subseteqq \Omega.$$

Moreover, we assume that ρ satisfies the following conditions:

(i) $\rho(w) \in [0,1]$ for all $w \in \Omega$;

(ii) $M_\rho = \lambda(\{w \in \Omega \mid \rho(w) = 1\}) > 0$.

For the parameter t running in the open interval]0,1[define the families of functions

$$\begin{cases} \rho_t : \mathbb{T} \ni w \leadsto \dfrac{t\rho(w)}{1-t\rho(w)} \in \mathbb{R}_+; \\ \varphi_t = 1+\rho_t = \dfrac{1}{1-t\rho}; \\ \Theta_t = \log \varphi_t. \end{cases}$$

Then $\Theta_t \in \mathscr{C}^2(\mathbb{T})$ and the Cauchy transform of Θ_t (where $C_z: w \longmapsto \frac{1}{1-w\bar{z}}$ denotes the Cauchy-Szegö kernel of D at $z \in D$)

$$h_t: D \ni z \longmapsto \langle\Theta_t, C_z\rangle \in \mathbb{C}$$

belongs to $\mathcal{H}^\infty(D)$ for all values of the parameter $t \in]0,1[$.

As a first result we state

Theorem 2. For all $t \in]0,1[$ the boundary value functions $\tilde{h}_t$ associated with the functions $h_t \in \mathcal{H}^\infty(D)$ satisfy the smoothness conditions

$$\mathrm{Re}\ \tilde{h}_t \in \mathscr{C}^2(\mathbb{T}), \quad \mathrm{Im}\ \tilde{h}_t \in \mathscr{C}^1(\mathbb{T}).$$

The proof follows by standard arguments based upon Abel-Poisson means and conjugate Abel-Poisson means of the functions h_t.

After this preparation we define the functions

$$\psi_t = e^{-\frac{1}{2}h_t} \qquad (t \in]0,1[).$$

Then $\psi_t \in \mathcal{H}^\infty(D)$ and we may add the following result:

Theorem 3. The functions $\mathrm{Re}\ \tilde{\psi}_t$ and $\mathrm{Im}\ \tilde{\psi}_t$ are multipliers of the space $\mathrm{BMO}(\mathbb{T})$, i.e., the following inclusions

$$(\mathrm{Re}\ \tilde{\psi}_t) \cdot \mathrm{BMO}(\mathbb{T}) \hookrightarrow \mathrm{BMO}(\mathbb{T}),$$

$$(\mathrm{Im}\ \tilde{\psi}_t) \cdot \mathrm{BMO}(\mathbb{T}) \hookrightarrow \mathrm{BMO}(\mathbb{T}),$$

hold for $t \in]0,1[$.

The preceding theorem combined with the Fefferman-Stein duality mentioned above furnishes continuous Toeplitz operators that are suitable to replace the M. Riesz projection theorem in the case $p = 1$.

Theorem 4. The Toeplitz operators $P_{\tilde{\psi}_t}$ and $P_{\bar{\tilde{\psi}}_t}$ with symbols $\tilde{\psi}_t$ and $\bar{\tilde{\psi}}_t (t \in]0,1[)$, respectively, are continuous endomorphisms of the Hardy subspace $\mathcal{H}^1(\mathbb{T})$ of $L^1(\mathbb{T})$ with respect to its norm topology. Moreover, we have

$$\sup_{t \in]0,1[} ||P_{\tilde{\psi}_t}||<+\infty$$

and

$$\sup_{t \in]0,1[} ||P_{\bar{\tilde{\psi}}_t}||<+\infty.$$

For details of the proof the reader is referred to the paper [5].

In view of the fact that

$$\tilde{\psi}_t \in \mathcal{H}^\infty(\mathbb{T}), \ \frac{1}{\tilde{\psi}_t} \in \mathcal{H}^\infty(\mathbb{T})$$

and

$$\varphi_t = \frac{1}{|\tilde{\psi}_t|^2}$$

for $t \in]0,1[$, the preceding result yields by an application of the inverse mapping theorem the following result:

Theorem 5. The family of Toeplitz operators $(P_{\varphi_t})_{t \in]0,1[}$ form topological automorphisms of $\mathcal{H}^1(\mathbb{T})$ with inverses

$$P_{\varphi_t}^{-1} = P_{\tilde{\psi}_t} \circ P_{\bar{\tilde{\psi}}_t}.$$

Moreover, the family of operators $(P_{\varphi_t}^{-1})_{t \in]0,1[}$ is uniformly bounded on the Hardy subspace $\mathcal{H}^1(\mathbb{T})$ of $L^1(\mathbb{T})$, i.e., we have

$$\sup_{t \in]0,1[} ||P_{\varphi_t}^{-1}||<+\infty.$$

5. Approximations with respect to the Hardy norm $\|.\|_{\mathcal{H}^1(D)}$. Retain the notations introduced in Section 4. Based on the preceding results we are now in a position to construct the announced approximation process for functions $f \in \mathcal{H}^1(D)$ by using only local boundary values of f.

As a first step an application of Harnack's inequality for the unit disc D yields the estimate

$$|\psi(z)| \leq (\sqrt{1-t})^{M_\rho \frac{1-|z|}{1+|z|}}$$

for $z \in D$ and $t \in]0,1[$. In view of the hypothesis $M_\rho > 0$ (see (ii) supra), the family $(\psi_t)_{t \in]0,1[}$ converges pointwise in D towards the null function as $t \to 1-$. Notice that an application of the operator $P_{\varphi_t}^{-1}$ to the Cauchy-Szegö kernel C_z of D furnishes

$$\begin{aligned} P_{\varphi_t}^{-1}(C_z) &= P_{\tilde{\psi}_t} \circ P_{\overline{\tilde{\psi}}_t}(C_z) \\ &= \overline{\psi_t(z)}\tilde{\psi}_t C_z \end{aligned}$$

for $z \in D$ and $t \in]0,1[$. The pointwise estimate above combined with the bound

$$\sup_{t \in]0,1[} \|\tilde{\psi}_t\|_\infty \leq 1$$

gives

$$\lim_{t \to 1-} \|P_{\varphi_t}^{-1}(C_z)\|_1 = 0$$

for $z \in D$. Theorem 5 and the standard fact that $\{C_z \mid z \in D\}$ forms a total subset of $\mathcal{H}^1(\mathbb{T})$ with respect to the norm topology imply

$$\lim_{t \to 1-} \|P_{\varphi_t}^{-1} f\|_1 = 0$$

for all functions $f \in \mathcal{H}^1(\mathbb{T})$. Moreover, the identities

$$P_1 = P_{\varphi_t}^{-1} \circ P_{\varphi_t} = P_{\varphi_t}^{-1} \circ (P_1 + P_{\rho_t})$$

imply

$$P_{\varphi_t}^{-1} = P_1 - P_{\varphi_t}^{-1} \circ P_{\rho_t}$$

for $t \in]0,1[$. This establishes our main result.

Theorem 6. Let the function $f \in \mathcal{H}^1(D)$ be given. Starting with the function $\rho \in \mathscr{C}^2(\mathbb{T})$ which satisfies the condition (i) and (ii) of Section 4 supra, construct the families of functions $(\rho_t)_{t \in]0,1[}$ and $(\psi_t)_{t \in]0,1[}$ via the prescriptions mentioned above. For $t \in]0,1[$ define the functions

$$f_t : D \ni z \longrightarrow \psi_t(z) \int_\Omega \frac{\bar{\tilde{\psi}}_t(w)\rho_t(w)\tilde{f}(w)}{1-\bar{w}z} d\lambda(w) = \psi_t(z)\langle \tilde{f}, \rho_t \tilde{\psi}_t C_z \rangle$$

- then $f_t \in \mathcal{H}^1(D)$ for $t \in]0,1[$ and

$$\lim_{t \to 1-} \|f - f_t\|_{\mathcal{H}^1(D)} = 0$$

holds.

Corollary (Szegö). Let $f \in \mathcal{H}^1(D)$ - then $\tilde{f}|\Omega$ implies $f=0$.

6. Summary. In the preceding sections the problem of constructing explicitly approximations in the "biggest" normed Hardy space $\mathcal{H}^1(D)$ from local boundary data is solved by using Toeplitz operators with $\mathscr{C}^2$-symbols on $\mathbb{T}$ smooth enough to multiply the strong dual BMO($\mathbb{T}$) of the Hardy subspace $\mathcal{H}^1(\mathbb{T})$ of $L^1(\mathbb{T})$. For additional material concerning the recovering problem, see Zarantonello [7] and the recent paper by Zayed [8] which both deal with certain modifications of the Patil technique.

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