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CHARACTERIZATIONS OF SPACES OF SOBOLEV TYPE WITH A DOMINATING MIXED DERIVATIVE BY DIFFERENCES

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1. Introduction. Let $0<p<\infty$ and let R_2 be the real plane. Its points are denoted by $x=(x_1,x_2)$. As usual $L_p=L_p(R_2)$ is defined as the space of all measurable functions f such that

$$\|f\,|\,L_p\| = \left(\int_{R_2} |f(x)|^p\,dx\right)^{\frac{1}{p}} < \infty \tag{1}$$

(dx Lebesgue measure). If $\bar{l}=(l_1,l_2)$ is a pair of non-negative in-tegers and if $1<p<\infty$ then we put

$$S_p^{\bar{l}}W = \Big\{ f \,|\, f\in L_p,\ \|f\,|\,S_p^{\bar{l}}W\| = \tag{2}$$
$$= \|f\,|\,L_p\| + \Big\|\frac{\partial^{l_1}}{\partial x_1^{l_1}} f\,\Big|\,L_p\Big\| + \Big\|\frac{\partial^{l_2}}{\partial x_2^{l_2}} f\,\Big|\,L_p\Big\| + \Big\|\frac{\partial^{l_1+l_2}}{\partial x_1^{l_1}\partial x_2^{l_2}} f\,\Big|\,L_p\Big\| < \infty \Big\}$$

The spaces $S_p^{\bar{l}}W$ are the well-known Sobolev spaces with a dominating mixed derivative as introduced and investigated by P. I. Lizorkin, S. M. Nikol'skij [5]. For the sake of simplicity we restrict ourselves to spaces on R_2. The progress in Fourier analysis during the last 15 years enables us to deal with a lot of classical function spaces such as Besov spaces (=Lipschitz spaces), Sobolev spaces, Lebesgue spaces (=Besselpotential spaces = Liouville spaces), Hölder spaces, Zygmund spaces, Hardy spaces, and the space BMO from a unified point of view. For a systematic treatment of isotropic spaces we refer to H. Triebel [13]. Smoothness properties of functions and distributions are described by means of appropriate decompositions in entire analytic functions. In particular the classical spaces can be extended to values $0<p<1$. However, the construction looks somewhat complicated. Thus, the question arises wether the new spaces can be characterized by classical methods, e. g. by means of derivatives and differences or by means of approximation, too. For isotropic spaces the problem was solved by H. Triebel [13, 2.5 and 2.12]. The anisotropic case was partly treated by Ф. A. Kaljabin [3,4]. The spaces $S_p^{\bar{I}}W$ are contained in the scale $S_{p,q}^{\bar{r}}F$, where $0 < p < \infty$, and $\bar{r} = (r_1, r_2)$ with $-\infty < r_i < \infty$ $(i=1,2)$. For the definition see section 2. The aim of this paper is to state some results which show that the spaces $S_{p,q}^{\bar{r}}F$ (and hence the spaces $S_p^{\bar{I}}W$) can be characterized by mixed differences provided that r_1 and r_2 are large enough. This will be done in section 3. Finally in section 4 we shall add some remarks concerning other types of spaces with dominating mixed smoothness properties.

2. The spaces $S_{p,q}^{\bar{r}}F$. Let Φ be the class of sequences $\varphi = \{\varphi_j(t)\}_{j=0}^{\infty}$ of infinitely differentiable functions on the real axis such that

(i) $\operatorname{supp} \varphi_0 \subset [-2,2]$,

$\operatorname{supp} \varphi_j \subset [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}]$; $j = 1, 2, \dots$,

(ii) $|\varphi_j^{(m)}(t)| \le c_m 2^{-mj}$; $m = 0, 1, \dots$; $j = 0, 1, \dots$; $t \in R_1$,

(iii) $$\sum_{j=0}^{\infty} \varphi_j(t) = 1, \quad t \in R_1.$$

Systems of test functions of this type are standard in the modern theory of function spaces. Of course, Φ is not empty. Further we shall denote by $S' = S'(R_2)$ the space of tempered distributions and by F, F^{-1} the Fourier transform and its inverse in S'. L_p has the meaning of (1).

Definition. Let $0 < p < \infty$, $0 < q \le \infty$, and let $\bar{r} = (r_1, r_2)$ with $-\infty < r_1 < \infty$, and $-\infty < r_2 < \infty$. Let $\varphi = \{\varphi_j(t)\}_{j=0}^{\infty} \in \Phi$. Then

$$S_{p,q}^{\bar{r}}F = \{ f \mid f \in S', \ \| f \mid S_{p,q}^{\bar{r}}F \|^{\varphi} = \tag{3}$$

$$= \Big\| \Big(\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} 2^{jr_1 q + kr_2 q} \big| \big(F^{-1}(\varphi_j(\xi_1)\varphi_k(\xi_2)Ff)\big)(x)\big|^q \Big)^{\frac{1}{q}} \Big| L_p \Big\| < \infty \}$$

(with the usual modification if $q = \infty$).

The spaces $S_{p,q}^{\bar{r}}F$ are due to H. Triebel [11, V, Definition 2.4/2] (or [12, 2.5.3]). We call them spaces of Triebel-Lizorkin type with dominating mixed smoothness properties. The spaces $S_{p,q}^{\bar{r}}F$

do not depend on $\varphi \in \Phi$, all quasinorms $\| \cdot | S_{p,q}^{\bar{r}} F \|^{\varphi}$ are equivalent to each other. Equipped with $\| \cdot | S_{p,q}^{\bar{r}} F \|^{\varphi}$, $\varphi \in \Phi$, $S_{p,q}^{\bar{r}} F$ becomes a quasi-Banach space (Banach space if $\min(p,q) \geq 1$).

Proposition. Let $1 < p < \infty$.

(i) If $\bar{r} = (r_1, r_2)$ with $-\infty < r_1 < \infty$ and $-\infty < r_2 < \infty$, then

$$\begin{aligned} S_{p,2}^{\bar{r}} F = S_p^{\bar{r}} H = \{ f \mid f \in S', \ \| f \mid S_p^{\bar{r}} H \| \\ = \| F^{-1} [(1+\xi_1^2)^{\frac{r_1}{2}} (1+\xi_2^2)^{\frac{r_2}{2}} F f] \mid L_p \| < \infty \} \end{aligned} \tag{4}$$

(ii) If $\bar{l} = (l_1, l_2)$ with non-negative integers l_i $(i=1,2)$ then

$$S_{p,2}^{\bar{l}} F = S_p^{\bar{l}} W . \tag{5}$$

A Proof can be found in H. Triebel [11, III]. It shows that the classical Sobolev-Lebesgue spaces with a dominating mixed derivative of P. I. Lizorkin, S. M. Nikol'skij [5] are special cases of $S_{p,q}^{r} F$. Further properties can be found in [11,7 - 12].

3. The main theorem. We shall give an equivalent characterization of $S_{p,q}^{r} F$ by means of differences provided that r_1 and r_2 are large enough. Let us start with some notations. If $f(x) = f(x_1, x_2)$ is a function defined on R_2 and if $\bar{m} = (m_1, m_2)$ with $m_i = 1, 2, \ldots$ $(i=1,2)$ then we put

$$\Delta_{h_1,1}^1 f(x) = f(x_1+h_1, x_2) - f(x), \quad \Delta_{h_2,2}^1 f(x) = f(x_1, x_2+h_2) - f(x),$$

$$\Delta_{h_i,i}^{m_i} f(x) = \Delta_{h_i,i}^1 (\Delta_{h_i,i}^{m_i-1} f)(x) \ ; \quad i = 1, 2,$$

$$\Delta_h^{\bar{m}} f(x) = \Delta_{h_2,2}^{m_2} (\Delta_{h_1,1}^{m_1} f)(x), \quad h = (h_1, h_2) \in R_2 .$$

Theorem. Let $0 < p < \infty$, $0 < q \leq \infty$ and let $\bar{r} = (r_1, r_2)$ with $0 < r_1 < \infty$ and $0 < r_2 < \infty$. Let $\bar{m} = (m_1, m_2)$ be a pair of natural numbers. If

$$m_1 > r_1 > \frac{1}{\min(p,q)} \quad \text{and} \quad m_2 > r_2 > \frac{1}{\min(p,q)}$$

then

$$\| f \mid S_{p,q}^{\bar{r}} F \|_{\bar{m}} = \| f \mid L_p \| +$$

$$+ \| (\int_{R_1} |h_1|^{-r_1 q} | \Delta^{m_1}_{h_1,1} f(x)|^q \frac{dh_1}{|h_1|})^{\frac{1}{q}} | L_p \|$$

$$+ \| (\int_{R_1} |h_2|^{-r_2 q} | \Delta^{m_2}_{h_2,2} f(x)|^q \frac{dh_2}{|h_2|})^{\frac{1}{q}} | L_p \| \qquad (6)$$

$$+ \| (\int_{R_2} |h_1|^{-r_1 q} |h_2|^{-r_2 q} | \Delta^{\overline{m}}_{h} f(x)|^q \frac{dh_1}{|h_1|}\frac{dh_2}{|h_2|})^{\frac{1}{q}} | L_p \|$$

is an equivalent quasi-norm in $S^{\overline{r}}_{p,q}F$ (with the usual modification if $q = \infty$).

The proof of the theorem uses methods of Fourier analysis such as inequalities for maximal functions as well as ideas of S. M. Nikol'skij [6]. We refer to [10]. By means of (4) and (5) we obtain new characterizations for the classical spaces $S^{\overline{r}}_pW$ and $S^{\overline{r}}_pH$. Moreover, note that also the spaces with values p, $0 < p < 1$ allow characterizations of that type. For isotropic spaces this fact is well-known (cf. [13 , 2.5.10].).

4. Remarks. Let us add few remarks concerning related spaces and analogous characterizations.

(i) The spaces $S^{\overline{r}}_{p,q}F$ are related to the spaces $S^{\overline{r}}_{p,q}B$ of S. M. Nikol'skij [6] and T. I. Amanov [1,2] which have been introduced via differences for values $1 \le p \le \infty$ and $1 \le q \le \infty$. We can extend these spaces to values $0 < p < 1$ by changing the order of summation and integration in (3). Then an analogous assertion as in our theorem holds true. In (6) we have to change the order of integration with respect to h and x. For the details see [10].

(ii) The spaces $S^{\overline{r}}_{p,q}F$ and $S^{\overline{r}}_{p,q}B$ can be extended to mixed quasi-norms replacing the quasi-norms in l_q and L_p appearing in (3) by those ones in $l_{\overline{q}}$ and $L_{\overline{p}}$, $\overline{p} = (p_1, p_2)$, $\overline{q} = (q_1, q_2)$, respectively. Then all assertions from the unmixed case as above and in (i) can be carried over (with modifications concerning the conditions with respect to $\overline{m}$ and $\overline{r}$). We refer to [10].

(iii) In [7] and [8] we considered a third scale of spaces with dominating mixed smoothness properties, $SB^{\overline{r}}_{\overline{p},\overline{q}}$ with $\overline{p} = (p_1, p_2)$, $\overline{q} = (q_1, q_2)$ and $\overline{r} = (r_1, r_2)$, which is related to the spaces $B^{r_2}_{p_2,q_2}(R_1, B^{r_1}_{p_1,q_1}(R_1))$ (vector-valued Besov-spaces). Analogous results as in our theorem can be proved (cf. [10]).

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