КОНСТРУКТИВНАЯ ТЕОРИЯ
ФУНКЦИЙ'84
София, 1984, с. 803—807

CONSTRUCTIVE THEORY
OF FUNCTIONS'84
Sofia, 1984, p.803-807

ON COMPLEX SPLINE INTERPOLATION ON THE UNIT CIRLE

Gunther Schmidt

In this note we prove the existence of complex polynomial splines of arbitrary odd degree with arbitrarily spaced knots on the unit cirlce, which interpolate to prescribed values at these knots. The interpolatory properties of complex splines on uniform meshes have been investigated in several papers ([2],[5],[7],[8]), whereas the case of nonuniform meshes was considered only for cubic splines ([1]). Our proof uses an equivalence relation from [3], which gives simple proofs for splineapproximation methods of pseudodifferential equations ([3],[6]) and periodic spline interpolation ([4]), too.

Let $\Delta = \{x_j\}_{-\infty}^{+\infty}$ be a 1-periodic mesh on the real line, i.e. $x_j < x_{j+1}$, $x_{j+n} = x_j + 1$ for some $n \in \mathbb{N}$ and all $j \in \mathbb{Z}$. By $C_r(\Delta)$, $r \in \mathbb{N}$, we denote the class of complex-valued, 1-periodic and $(r-1)$-times continuously differentiable functions on $\mathbb{R}$, which agree in values with a polynomial in $z(x) = e^{2\pi i x}$, $x \in \mathbb{R}$, of degree at most r on each interval (x_j, x_{j+1}), $j \in \mathbb{Z}$. $C_0(\Delta)$ denotes the class of 1-periodic piecewise constant functions with respect to Δ. Obviously $C_r(\Delta)$ can be considered as the class of complex polynomial splines of degree r on the unit circle subordinate to the partition $\left\{z_j = e^{2\pi i x_j}\right\}_{j=1}^{n}$. It is well known that $\dim C_r(\Delta) = \max\{n, r+1\}$.

By H^s, $s \in \mathbb{R}$, we denote the periodic Sobolev space of order s, i.e. the closure of all 1-periodic C^∞ functions with respect to the norm

$$\|f\|_s = \left\{ |\hat{f}_0|^2 + \sum_{0 \neq k \in \mathbb{Z}} |\hat{f}_k|^2 |2\pi k|^{2s} \right\}^{1/2} ,$$

where $\hat{f}_k = \int_0^1 f(x) z^{-k}(x)\, dx$. Clearly $H^0 = L_2$, by $(\cdot,\cdot)$ we denote the scalar product in L_2. By setting

(1) $$Df(x) := \hat{f}_0 + \sum_{0 \neq k \in \mathbb{Z}} \hat{f}_k (2\pi k)\, z^k(x)$$

we define an isometric isomorphism $D: H^s \longrightarrow H^{s-1}$, $s \in \mathbb{R}$. Furthermore we define two onedimensional operators J and J_Δ by

$$Jf(x) = \int_0^1 f(y) dy \quad \text{and} \quad J_\Delta f(x) = \sum_{j=1}^{n} f(x_j)(x_{j+1} - x_{j-1})/2 \, .$$

Note that, for sufficiently smooth periodic functions f, $Df = \frac{1}{i}\frac{df}{dx} + Jf$. From $J_\Delta^2 = JJ_\Delta = J_\Delta$ and $J^2 = J_\Delta J = J$ we conclude $(I - J + J_\Delta)^{-1} = I + J - J_\Delta$.

Lemma 1 [3]. For $f \in H^s$, $s > 1/2$, the equations $f(x_j) = 0$, $j=1,\dots,n$, hold if and only if

(2) $$(D(I-J+J_\Delta)f, v) = 0 \quad \text{for all} \quad v \in C_0(\Delta).$$

Proof. Let $f \in C^1$. Then $D(I-J+J_\Delta)f = \frac{1}{i} f' + J_\Delta f$ and

$$(D(I-J+J_\Delta)f, v) = \frac{1}{i} \int_{x_0}^{x_n} f'(x)\overline{v(x)}\, dx + J_\Delta f \int_{x_0}^{x_n} \overline{v(x)}\, dx.$$

Denoting $h_j = x_j - x_{j-1}$ and

$$v_j(x) = \begin{cases} h_{j+1}^{-1}, & x \in [x_j, x_{j+1}), \\ -h_j^{-1}, & x \in [x_{j-1}, x_j), \\ 0, & \text{otherwise} \end{cases}, \quad j=1,\dots,n-1,$$

we have

$$(D(I-J+J_\Delta)f, v_j) = \frac{1}{i}\left[h_{j+1}^{-1}(f(x_{j+1}) - f(x_j)) - h_j^{-1}(f(x_j) - f(x_{j-1}))\right] = 0$$

if and only if there exists a constant c such that

$h_j^{-1}(f(x_j) - f(x_{j-1})) = c$. Hence, $f(x_n) = f(x_0) + c\sum_{j=1}^{n} h_j = f(x_0) + c$, by periodicity of f we obtain $c = 0$ and

(3) $\quad f(x_j) = \text{const}\ , \quad j=1,\dots,n.$

Since $C_0(\Delta)$ is spanned by the v_j and the constant function 1, we see that (2) holds if and only if (3) hold and, in addition,

$$(D(I-J+J_\Delta)f,1) = \frac{1}{i}\int_{x_0}^{x_n} f'(x)\,dx + J_\Delta f = f(x_0)\sum_{j=1}^{n}(h_{j+1}+h_j)/2 =$$
$$= f(x_0) = 0\ .$$

Therefore the assertion is true for $f \in C^1$. The case of $f \in H^s$, $s > 1/2$, follows from the embedding $H^s \subset C$ and the boundedness of the linear form $(\cdot, v)$ on H^s for any $v \in C_0(\Delta)$. ■

In order to apply this result, we seek a simple representation of $C_r(\Delta)$. Let $u \in C_r(\Delta)$. Then $(\frac{d}{dz})^r u \in C_0(\Delta)$ with the Fourier series

$$(\tfrac{d}{dz})^r u(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k\, k(k-1)\dots(k-r+1)\, z^{k-r}(x).$$

Therefore the mapping $(\frac{d}{dz})^r$ is an isomorphism between the spaces

$$\tilde{C}_r(\Delta) := \{u \in C_r(\Delta) : \hat{u}_k = 0\ ,\ k=0,\dots,r-1\}$$

and $\quad C_0^r(\Delta) := \{v \in C_0(\Delta) : \hat{v}_k = 0\ ,\ k=-r,\dots,-1\}$.

Since $\frac{d}{dz} = \frac{e^{-2\pi ix}}{2\pi i}\frac{d}{dx}$, we introduce the invertible mapping $\mathcal{D} := \frac{z^{-1}}{2\pi} D : H^s \to H^{s-1}$, $s \in \mathbb{R}$, where z denotes the operator of multiplication by the function $z(x)$. Obviously $\mathcal{D}^r = (\frac{d}{dz} + \frac{z^{-1}}{2\pi} J)^r$ maps $\tilde{C}_r(\Delta)$ isomorphically onto $C_0^r(\Delta)$ and, moreover, $\mathcal{D}^r z^k(x) = a_k z^{k-r}(x)$ with some $a_k \neq 0$. Hence we obtain $\operatorname{im}\mathcal{D}^r(C_r(\Delta)) = C_0^r(\Delta) \dotplus \operatorname{lin}(z^{-1}(x),\dots,z^{-r}(x))$. Thus we have proved

<u>Lemma 2.</u> $\quad C_r(\Delta) = \mathcal{D}^{-r}(C_0^r(\Delta) \dotplus \operatorname{lin}(z^{-1}(x),\dots,z^{-r}(x)))$.

Now we are in position to establish the existence of the complex spline interpolant on the unit circle.

<u>Theorem.</u> For any odd number $r \in \mathbb{N}$, any partition Δ with $n \geq r+1$

and any $y_j \in \mathbb{C}$, $j=1,\dots,n$, there exists a uniquely determined complex spline $u \in C_r(\Delta)$ such that

$$u(x_j) = y_j \quad , \quad j=1,\dots,n.$$

<u>Proof.</u> Let $u \in C_r(\Delta)$ with $u(x_j) = 0$, $j=1,\dots,n$. Obviously the assertion is proved when we show that $u = 0$. Since $z^{-r}(x_j)u(x_j)=0$, by Lemma 1 we conclude that $(D(I-J+J_\Delta)z^{-r}u,v) = 0$ for all $v \in C_0(\Delta)$. Evidently $z^r(I-J+J_\Delta)z^{-r}u \in C_r(\Delta)$, such that, by Lemma 2, $z^r(I-J+J_\Delta)z^{-r}u = \mathcal{D}^{-r}w$ for some $w \in C_0^r(\Delta) \dotplus \mathrm{lin}(z^{-1}(x),\dots,z^{-r}(x))$. Hence

$$(4) \qquad (Dz^{-r}\mathcal{D}^{-r}w,v) = 0 \quad \text{for all} \quad v \in C_0(\Delta).$$

From (1) we have

$$\mathcal{D} f(x) = \sum_{k\in\mathbb{Z}} d_k \hat{f}_k z^{k-1}(x) \ , \text{ where } \ d_k = \begin{cases} k & , k \neq 0, \\ (2\pi)^{-1} & , k=0. \end{cases}$$

Hence, $\mathcal{D}^{-r}f(x) = \sum_{k\in\mathbb{Z}} (d_{k+1}\dots d_{k+r})^{-1}\hat{f}_k \, z^{k+r}(x)$, and, consequently,

$$(Dz^{-r}\mathcal{D}^{-r}w,v) = 2\pi \sum_{k\in\mathbb{Z}} d_k(d_{k+1}\dots d_{k+r})^{-1}\hat{w}_k\bar{\hat{v}}_k \ .$$

Since r is odd, we obtain $\lambda_k = 2\pi\, d_k(d_{k+1}\dots d_{k+r})^{-1} > 0$ for $k \neq -1,-3,\dots,-r$.

Let now $w(x) = s(x) + \sum_{k=-r}^{-1} a_k z^k(x) \neq 0$, $s(x) \in C_0^r(\Delta)$. If $s \neq 0$, then

$$(Dz^{-r}\mathcal{D}^{-r}w,s) = \sum_{\substack{k\in\mathbb{Z} \\ k\neq -1,\dots,-r}} \lambda_k |\hat{s}_k|^2 > 0 \ ,$$

i.e. (4) cannot be satisfied. Let $s = 0$. Since $\dim(C_0(\Delta)\setminus C_0^r(\Delta))=r$, there exists $v \in C_0(\Delta)\setminus C_0^r(\Delta)$ such that $(Dz^{-r}\mathcal{D}^{-r}w,v) = 1$. Thus (4) holds if and only if $w = 0$ and therefore

$$u = z^r(I+J-J_\Delta)z^{-r}\mathcal{D}^{-r}w = 0 \ . \ \blacksquare$$

References

[1] J.H. Ahlberg, E.N. Nilson, J.L. Walsh, Complex cubic splines. Trans.Amer.Math.Soc. <u>129</u> (1967), 321-413.

[2] J.H. Ahlberg, E.N. Nilson, J.L. Walsh, Properties of analytic splines.I: Complex polynomial splines. J.Math.Anal.Appl.

29 (1969), 262-278.

[3] D.N. Arnold, W.L. Wendland, On the asymptotic convergence of collocation methods. Math.Comp. 41 (1983), 349-381.

[4] J. Elschner, G. Schmidt, On spline interpolation in periodic Sobolev spaces. Inst.f.Math.d.AdWd.DDR, Preprint P-MATH-01/83.

[5] C.A. Micchelli, A. Sharma, Spline functions on the unit circle. Cardinal L-splines revisited. MRC.TSR №1918.

[6] G. Schmidt, The convergence of Galerkin and collocation methods with splines for pseudodifferential equations on closed curves. Zeitschr. Anal.und Anwend. 3, in print.

[7] I.J. Schoenberg, On polynomial spline functions on the unit circle, I, in "Proceedings of the conference on constructive theory of functions, Budapest", 403-418, Budapest 1972.

[8] J. Tzimbalario, Interpolation by complex splines. Trans.Amer. Math.Soc. 243 (1978), 213-222.

Akademie der Wissenschaften der DDR
Institut für Mathematik
DDR - 1086 Berlin
Mohrenstraße 39 , PSF 1304