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ON A THEOREM OF H.WHITNEY

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In his classical paper "On functions with bounded n^{th} differences" [1] , H.Whitney proved a conjecture of H.Burkill, and this is now a very popular and very important theorem in approximation theory with many applications in numerical analysis.

Let Ω be a set of real numbers and f is a function defined in Ω . We shall use the following notations:

$$\Delta_h^k f(x) := \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x+ih) ,$$

$$\omega_k(f;\Omega) := \sup\{|\Delta_h^k f(x)| : x, x+h, \dots, x+kh \in \Omega\} ,$$

$$\|f\|_\Omega := \sup\{|f(x)| : x \in \Omega\} ,$$

$$E_n(f;\Omega) := \inf\{\|f-P\| : P \in H_n\} ,$$

Where H_n is the set of the algebraic polynomials of a degree $\leq n$.

<u>Theorem 1</u>. (H.Whitney [1]). For $\Omega = [0,1]$, $[0,\infty)$, $(-\infty,\infty)$ and for every positive integer k there exists a constant $W_k(\Omega)$ depending only on k and Ω , such that for every function f defined and bounded on Ω , the following inequality holds:

$$E_{k-1}(f;\Omega) \leq W_k(\Omega)\, \omega_k(f;\Omega)$$

It is easy to see that

$$W_k([a,b]) = W_k([0,1))$$

for every $-\infty < a < b < \infty$.

In [1] H.Whitney has also proved the inequalities

$$W_k((-\infty,\infty)) \leq W_k([0,\infty)) \leq W_k([0,1]) ,$$

$$\frac{1}{2} \leq W_k([0,\infty)) \leq 1 ,$$

$$W_{2m}((-\infty,\infty)) = 1/\binom{2m}{m} ,$$

$$\frac{1}{2}\binom{2m}{m} \leqq W_{2m+1}((-\infty,\infty)) \leqq 1/\binom{2m+1}{m}$$

$$W_1([0,1]) = W_2([0,1]) = \frac{1}{2} ,$$

$$\frac{8}{15} \leqq W_3([0,1]) \leqq \frac{7}{10} ,$$

$$\frac{1}{2} \leqq W_4([0,1]) \leqq 3.2425 ,$$

$$\frac{1}{2} \leqq W_5([0,1]) \leqq 10.4 .$$

In the practical applications of the theorem 1, the values of the Whitney's constants $W_k([0,1])$ are important.

Until now, the best published estimation of $W_k([0,1])$ for every k which we know is [2]

$$W_k([0,1]) \leqq (k+1)k^k .$$

There are nonpublished announcements that

$$W_k([0,1]) \leqq 2^k ,$$

but we have not seen such a proof.

In [3] we prove

<u>Theorem 2</u>. Let $\Omega_m := \left\{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1\right\}$ and

$$A_k^m(p,q) := \inf\left\{\omega_k(f;\Omega_m) : \|f\|_{\Omega_m} = 1, f\left(\frac{p}{m}\right) = 1, f\left(\frac{q}{m}\right) = -1\right\} ,$$

then $W_k([0,1]) \leqq \left\{1/A_k^m(p_{i-1},p_i) : 0 = p_0 < \ldots < p_{i-1} < p_i < \ldots < p_k = m\right\}$.

It is easy to see that the numbers $A_k^m(p,q)$ may be calculated by linear programming technique.

Using theorem 2, we calculate the following estimations

$$W_4([0,1]) \leqq 1.26 \qquad \text{(for m=64)}$$

$$W_5([0,1]) \leqq 1.31 \qquad \text{(for m=80)}$$

$$W_6([0,1]) \leqq 1.67 \qquad \text{(for m=96)}$$

Using more powerfull computers, some better estimations may be found for bigger k .

Our conjecture [3] is that for every positive integer k , the inequality

$$W_k([0,1]) \leqq 1$$

holds.

All our attempts till now to prove these conjecture are not successful, and we shift to some generalizations of H.Whitney's theorem.

It is natural to ask for H.Whitney's theorem and H.Whitney's constant if Ω is different from $[0,1]$, $[0,\infty)$ and $(-\infty,\infty)$.

Definition 1. The set of real numbers Ω is A_k set if: 1) $|\Omega| \leqq k$, Ω has not more than k elements, or if 2) for every $\xi \in \Omega$ there exist $x, x+h, \dots, x+kh \in \Omega$ and $\xi \in \{x, x+h, \dots x+kh\}$.

Theorem 3. Necessary condition for $W_k(\Omega) < \infty$ is Ω to be A_k set.

Proof. Let Ω is not an A_k set and $\xi \in \Omega$ is such a point that there is not an arithmetic progression with k+1 element belonging to Ω and containing ξ . By the definition, the number of points in Ω are at least k+1. Let f_0 is a function defined on Ω and

$$f_0(x) = \begin{cases} 1 & for \quad x = \xi , \\ 0 & for \quad x \in \Omega \text{ and } x \neq \xi . \end{cases}$$

It is easy to see that $E_{k-1}(f_0;\Omega) > 0$. and

$\omega_k(f_0;\Omega) = 0$. □.

The condition in theorem 3 is not sufficient for $k \geqq 2$

Example. Let

$$\Omega'' = \{0, 2^{-3m}, 2^{-3m-1} : m = 0,1,2,\dots\}$$

and

$$f_1(x) = \begin{cases} -1 & for \quad x = 0 , \\ 0 & for \quad x = 2^{-3m-1} , \\ 1 & for \quad x = 2^{-3m} , \end{cases}$$

The set Ω'' is an A_2 set , $\omega_2(f_1;\Omega'')=0$ and $E_1(f;\Omega'')=1$

Definition 2. The set of real numbers Ω is B_k set if for every pair $x,y\in\Omega$, $h=y-x\neq 0$, the numbers $z=x+nh$, n-integer, belongs to Ω if $z\in[\inf\{\xi:\xi\in\Omega\},\sup\{\xi:\xi\in\Omega\}]$.

Theorem 4. Sufficient condition for $W_k(\Omega)<\infty$ is Ω to be B_k set.

Proof. If the set Ω has finite number of elements, then Ω is finite arithmetic progression and we may consider only the case $\Omega=\Omega_m$. In this case the proof of the theorem is very simple.

If the set Ω has infinite number of elements, then the proof is a slight modification of the proof in [4] . □.

For finding of estimates of $W_k(\Omega_m)$, the following lemma is usefull .

Lemma. Let

(1) $$\Delta^k_{1/m} f(p) = a_p \; ; \quad p=0,1,2,\dots,m-k \; .$$

then

(2) $$f(p)=\sum_{i=0}^{m-k}\left\{\binom{p-i-1}{k-1}_+ - \varphi_i(p)\right\}a_i \; ; \; p=0,1,2,\dots,m \; ,$$

where

$$\binom{s}{q}_+ = \begin{cases} 0 & \text{for } s<q\,, \\ \binom{s}{q} & \text{for } s\geqq q\,, \end{cases}$$

and $\varphi_i\in H_{k-1}$.

Proof. We check that the numbers (2) satisfy the linear equations (1)

$$\Delta^k_{1/m} f(p) = \Delta^k_{1/m}\sum_{i=0}^{m-k}\left\{\binom{p-i-1}{k-1}_+ - \varphi_i(p)\right\}a_i$$

$$= \sum_{i=0}^{m-k} a_i\,\Delta^k_{1/m}\binom{p-i-1}{k-1}_+ = a_p \, . \; □ \, .$$

Corollary.

$$W_k(\Omega_m) \leqq \sum_{i=0}^{m-k} E_{k-1}\left(\binom{p-i-1}{k-1}_+ ; \Omega_m\right) .$$

It is easy to see that:

$$W_k(\Omega_m) = 0 \quad \text{for} \quad m = 1, 2, 3, \ldots, k-1 ,$$

$$W_k(\Omega_k) = 2^{-k} ,$$

$$W_k(\Omega_{k+1}) = 1 / \binom{k}{[k/2]} \approx 2^{-k}\sqrt{\frac{\pi k}{2}} .$$

The observation of the numbers $W_k(\Omega_m)$ for small k inspires the conjecture that

$$W_k(\Omega_{qk}) \leqq W_k(\Omega_{qk+1}) \leqq \ldots \leqq W_k(\Omega_{(q+1)k-1})$$

and

$$W_k(\Omega_{qk-1}) \geqq W_k(\Omega_{qk}) .$$

The study of the asymptotics of $W_k(\Omega_m)$ for fixed k and $m \to \infty$ may help to estimate $W_k([0,1])$ because

$$\overline{\lim_{m\to\infty}}\, W_k(\Omega_m) \geqq 2\, W_k([0,1]) .$$

References

1. H.Whitney, On functions with bounded n^{th} differences, J.Math. Pure et Appl. 36, 1957, 67-95.

2. Бл. Сендов, Модифицированная функция стеклова, Докл. Болг.акад. наук, 36, 3, 1983, 315-317.

3. Bl.Sendov. On the constants of H.Whitney, C.R. Acad. Bulg.des Sci.,35, 4, 1982, 431-434.

4. Bl.Sendov. A new peoof of the H.Whitney's theorem, C.R.Acad. Bulg.des Sci., 35, 5, 1982, 609-611.

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