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APPROXIMATION AND INTERPOLATION BY TRANSCENDENTAL POLYNOMIALS

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1. Introduction. The aim of the paper is to prove the following two theorems. They have already been obtained in [3] by the same method. However the author hopes that their proofs presented here are more perspicuous than the former ones.

Let

(1) $$h(z) = c_0 + c_1 z + c_2 z^2 + \dots , \quad z \in \mathbb{C},$$

be a transcendental entire function. Let E be a compact subset of the complex plane $\mathbb{C}$ such that $\mathbb{C}-E$ is connected.

THEOREM 1. For every function f holomorphic in a neighborhood of E there exists a sequence of polynomials of two complex variables $\{P_n(z,w)\}$ such that $\deg P_n \leq n$ and

(2) $$\lim_{n\to\infty} \left(\sup_{z\in E} |f(z) - P_n(z,h(z))| \right)^{1/n} = 0 .$$

It is known [4] that if the logarithmic capacity of E vanishes then for every function f holomorphic in a neighborhood of E there exists a sequence of polynomials of a complex variable $\{p_n\}$ such that $\deg p_n \leq n$ and

$$\lim_{n\to\infty} \left(\sup_{z\in E} |f(z) - p_n(z)| \right)^{1/n} = 0.$$

It follows that Theorem 1 is interesting only if the capacity of E is positive. Put

$$\rho_n(f,E) := \inf\{\|f-p\|_E \, ; \, p(z) = a_0+a_1z+\dots+a_nz^n\}$$

and let T_n be a polynomial of degree $\leq n$ such that $\rho_n(f,E) = \|f-T_n\|_E$. It is known [4] that if cap $E > 0$, then

(3) $$\lim_{n} \sqrt[n]{\rho_n(f,E)} = 0 ,$$

if and only if there exists an entire function $\tilde{f}$ with $\tilde{f} = f$ on E . Therefore, if cap $E > 0$ and if f is a holomorphic function in a neighborhood of E which cannot be extended to an entire function, then $\limsup_{n\to\infty} \sqrt[n]{\|f-T_n\|_E} > 0$, while the best approximation by "transcendental"polynomials $P_n(z,h(z))$ satisfies (2) .

THEOREM 2. Let h be a transcendental entire function on $\mathbb{C}$. Let E be an infinite subset of $\mathbb{C}$ with connected complement $\mathbb{C}-E$. Then for every $n \geq 1$ there exists $m_n := \binom{n+2}{2}$ points $z_{n1}, \ldots, z_{nm_n}$ in E such that for every function f holomorphic in a neighborhood of E there exists a unique polynomial $L_n(z,w)$ of two complex variables with the following properties

(i) $\deg L_n \leq n$ and $L_n(z_{nj}, h(z_{nj})) = f(z_{nj})$ $(j=1,\ldots,m_n)$,

(ii) $\lim_{n\to\infty} \left(\sup_{z\in E} |f(z) - L_n(z,h(z))| \right)^{1/n} = 0$.

The points $z_{nj} \in E$ $(j=1,\ldots,m_n)$ and the interpolating polynomials $L_n(z,w)$ satisfying (i) and (ii) can be constructed as follows.

Let $e_1, e_2, \ldots$ be a sequence composed of all monomials $z^k w^l$ of two complex variables $(z,w) \in \mathbb{C}^2$ such that $\deg e_n \leq \deg e_{n+1}$, e.g.

$$\{e_n\} = \{1, z, w, z^2, zw, w^2, \ldots, z^k, z^{k-1}w, \ldots, w^k, \ldots\},$$

so that $e_1(z,w) = 1$, $e_2(z,w) = z$, $e_3(z,w) = w$, etc.

Given any system of n points $p_1,\ldots,p_n$ of $\mathbb{C}^2$, we define the generalized Vandermondian V of order n by

$$(4) \qquad V(p_1,\ldots,p_n) := \det\left[e_i(p_j)\right]_{i,j=1,\ldots,n}.$$

A system of points $q_{n1},\ldots,q_{nn}$ (or shortly $q_1,\ldots,q_n$) of a compact subset K of $\mathbb{C}^2$ is called a system of the extremal points of K with respect to V of order n, if

$$(5) \qquad |V(q_1,\ldots,q_n)| = \max\left\{ |V(p_1,\ldots,p_n)| \; ; \; (p_1,\ldots,p_n) \in K^n \right\}.$$

Put $K := \{(z,h(z)) \; ; \; z \in E\}$ and $z_{nj} := \pi(q_j)$ $(j=1,\ldots,m_n)$, where $q_1,\ldots,q_{m_n}$ is a system of the extremal points of K with respect to V of order m_n, and π denotes the natural projection of $\mathbb{C}^2$ onto the first coordinate plane. Since E is infinite and h is transcendental, one can easily show that

$$V(q_1,\ldots,q_{m_n}) \neq 0, \quad n \geq 1.$$

The number of monomials $z^k w^l$ with $k+l \leq n$ is equal m_n. Therefore by the Cramer's formula the function $L_n(z,w) \equiv L_n(z,w;f)$ defined by

$$(6) \qquad L_n(z,w;f) := \sum_{j=1}^{m_n} f(z_{nj}) L_n^{(j)}(z,w),$$

where

(7) $$L_n^{(j)}(z,w) := \frac{V(q_1,\dots,q_{j-1},p,q_{j+1},\dots,q_{m_n})}{V(q_1,\dots,q_{m_n})}\ , \quad p:=(z,w)\in\mathbb{C}^2 ,$$

is the unique polynomial satisfying (i).

The definition of the extremal points $q_1,\dots,q_{m_n}$ of K implies that

$$|L_n^{(j)}(z,w)| \leq 1 \quad \text{on } K \quad \text{for } j = 1,\dots,m_n .$$

If $P_n(z,w)$ is a polynomial of degree $\leq n$, then

$$f(z) - L_n(z,w;f) \equiv f(z) - P_n(z,w) + L_n(z,w;P_n-f) .$$

Thus

$$\sup_{z\in E} |f(z)-L_n(z,h(z);f)| \leq (1+m_n)\sup_{z\in E} |f(z)-P_n(z,h(z))| .$$

Hence, if P_n are polynomials satisfying (2), we see that the interpolating polynomials $L_n(z,w) := L_n(z,w;f)$ satisfy (i) and (ii).

The proof of Theorem 1 is obtained in the next section by the method of the extremal function Φ_K. The function Φ_K is defined for every compact subset of $\mathbb{C}^N$ $(N\geq 1)$ by the formula

(8) $$\Phi_K(z) := \sup_{n\geq 1}(\sup\{|P(z)| ;\ P(z) = \sum_{|\alpha|\leq n} c_\alpha z^\alpha ,\ \|P\|_K = 1\})^{1/n} ,\quad z\in\mathbb{C}^N,$$

where $\alpha\in Z_+^N$ and $|\alpha| := \alpha_1+\dots+\alpha_N$ (see [2]).

2. Proof of Theorem 2. We need the following Lemmas.

LEMMA 1 (Lemma 8.4 in [2]). Let P_j $(j=1,\dots,m)$ be a polynomial of N complex variables with $\deg P_j \leq d$. Given $R>1$, put

(9) $$D_s := \{\varphi(z) < R^s\} ,\quad 0<s\leq 1 ,$$

where $\varphi := (\sup_{1\leq j\leq m} |P_j|)^{1/d}$. If the polynomial polyhedron $D := D_1$ is bounded, then for every function f holomorphic in D

(10) $$\rho(f,\overline{D}_s) \leq R^{s-1} \quad \text{for all } s \text{ with } 0<s<1 ,$$

where $\rho(f,\overline{D}_s) := \limsup_{n\to\infty} \sqrt[n]{\rho_n(f,\overline{D}_s)}$ with $\rho_n(f,\overline{D}_s) := \inf\{\|f-P\|_{\overline{D}_s} ;\ \deg P \leq n\}$.

LEMMA 2. If $K := \{(z,h(z)) ;\ z\in E\}$, then

(11) $$\Phi_K(z,w) = +\infty \quad \text{in } \mathbb{C}^2-K .$$

P r o o f. Put $A := \{(z,h(z));\ z\in\mathbb{C}\}$ and let (a,b) be a fixed point of $\mathbb{C}^2-A$. Put $P_n(z,w) := w - h_n(z)$, where $h_n(z) = c_0+c_1z+\dots+c_nz^n$ is the n-th partial sum of the series (1). Then $\|P_n\|_K^{1/n} = \|h - h_n\|_E^{1/n} \to 0$ as $n\to\infty$. Hence

$$|P_n(z,w)|^{1/n} = \|P_n\|_K^{1/n}\,\Phi_K(z,w) \quad \text{in } \mathbb{C}^2 ;$$

in particular

$$|b - h_n(a)|^{1/n} = |P_n(a,b)|^{1/n} \leqslant \|P_n\|_K^{1/n} \Phi_K(a,b), \quad n \geqslant 1 .$$

In the limit as $n \to \infty$, we get

$$1 \leqslant 0 \cdot \Phi_K(a,b),$$

which implies that $\Phi_K(a,b) = +\infty$.

Now one can prove (for details see the proof of Proposition 3.1 in [3]) that the function $u(z) := \log \Phi_K(z,h(z))$ is in $\mathbb{C}-E$ locally a limit of an increasing sequence of harmonic functions. Hence by the Harnack's principle u is either harmonic or $u \equiv +\infty$ in $\mathbb{C}-E$. The function h being transcendental we get by Sadullaev [1] that u is not locally bounded in $\mathbb{C}$. Therefore $u = +\infty$ in $\mathbb{C}-E$ or equivalently $\Phi_K = +\infty$ in $\mathbb{C}^2-A$.

Now we are in a position to prove Theorem 1. Namely, since f is holomorphic in a neighborhood of E, we can assume that the function $\tilde{f}(z,w) := f(z)$ is holomorphic in a bounded open neighborhood Ω of K. By Lemma 2 , given any fixed $R>1$, we can find a positive integer d and a finite number of polynomials $P_j(z,w)$ $(j=1,\ldots,n)$ of degree $\leqslant d$ such that

$$K \subset D_s \subset \Omega, \quad 0 < s \leqslant 1,$$

where D_s is the polynomial polyhedron defined by the formula (9). By Lemma 1

$$\rho(f,K) \leqslant \rho(f,\bar{D}_s) \leqslant R^{s-1} \quad \text{for all s with} \quad 0<s<1,$$

which implies that $\rho(f,K) \leqslant 1/R$. By the arbitrariness of R we get $\rho(f,K) = 0$, which is equivalent to the claim of Theorem 1 .

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