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L_1-APPROXIMATION WITH BLENDING FUNCTIONS

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1. Introduction. A recent survey by Cheney [1] gives us an introduction to 'approximation of multivariate functions by combinations of univariate ones'. We discuss L_1-approximation with blending functions:

Let $C_1^p(I)$ denote the space of p times continuously differentiable functions, whose p-th derivatives are integrable (we always use Lebesgue measure in this paper) on the finite interval $I \subset \mathbb{R}$. Let $U \subset C_1^p(I)$, $V \subset C_1^q(J)$ be subspaces. We denote by

$$B^{p,q}(U,V) := U \otimes C_1^q(J) + C_1^p(I) \otimes V$$

the space of blending functions. We want to use ECT-systems (cf. Karlin-Studden [5]) to define the subspaces U and V. To this end, we recall some fundamental results on ECT-systems:

Let $\{u_1,\dots,u_m\} \subset C_1^m(I)$ be an ECT-system. We denote the Wronskian of $u_1,\dots,u_i$, $1 \le i \le m$, by $W_i := W(u_1,\dots,u_i)$ and (by a proper choice of signs and $W_o := 1$) the positive generating functions by $\alpha_1 := u_1 = W_1$ and $\alpha_i := W_{i-1}W_{i+1}/W_i^2$, $2 \le i \le m$. Then we get for every $x_o \in I$ by

$$\begin{aligned} &\phi_1(x,x_o) := \alpha_1(x), \\ (1)\quad &\phi_i(x,x_o) := \alpha_1(x)\int_{x_o}^{x}\alpha_2(\xi_2)\int_{x_o}^{\xi_2}\dots\dots\int_{x_o}^{\xi_{i-1}}\alpha_i(\xi_i)d\xi_i\dots d\xi_2, \quad 2 \le i \le m, \end{aligned}$$

a fundamental solution of the differential equation ($f \in C_1^m(I)$)

$$W(u_1,\dots,u_m,f)/W_m =: (W_m/W_{m-1})\,\Delta_m f = 0$$

with $U_m := \operatorname{span}\{u_1,\dots,u_m\} = \ker \Delta_m$ (cf. Coppel [2], Karlin-Studden [5], Pólya [7]). The above differential equation possesses the decomposition

$$\Delta_i f(x) = D_i \circ D_{i-1} \circ \dots \circ D_1 f(x), \quad 1 \le i \le m,$$

with the first order differential operators $D_i f(x) := \frac{d}{dx}(f(x)/\alpha_i(x))$. Note, that for Δ_i the mean-value theorem by Pólya [7] holds true.

Using this notation and $\Delta_o := \mathrm{id}$ we get the Taylor series at $x_o \in I$ for $f \in C_1^m(I)$ by

$$(2) \qquad f(x) = \sum_{i=1}^{m} \frac{\Delta_{i-1} f(x_o)}{\alpha_i(x_o)} \phi_i(x,x_o) + \int_{x_o}^{x} \Delta_m f(s)\phi_m(x,s)ds$$

$$=: T_m(f,x_o)(x) + R_m(f,x_o)(x) \ ,$$

where $T_m(f,x_o) \in U_m$.

There is a general characterisation for L_1-proxima in the following framework (cf. Singer [9]):

Let $M \subset \mathbb{R}^r$ be measurable, $u^* \in U \subset L_1(M)$ and $f \in L_1(M) \setminus \overline{U}$. Then $\|f - u^*\|_1 \le \|f - u\|_1$ for all $u \in U$, iff there exists a $\sigma \in L_\infty(M) \simeq L_1(M)^*$ with the following properties:

$$(3) \qquad \|\sigma\|_\infty = 1 \ ,$$

$$(4) \qquad \int_M u\sigma = 0 \quad \text{for all } u \in U \ ,$$

$$(5) \qquad \sigma(x) = \text{sign}(f(x) - u^*(x)) \quad \text{for all } x \in M \setminus Z(f - u^*) \ ,$$

where $Z(g) := \{x \in M \mid g(x) = 0\}$ and $\text{sign}\, g(x) = g(x)/|g(x)|$ for $g(x) \neq 0$ and $\text{sign}\, g(x) = 0$ for $g(x) = 0$. We call functions $\sigma \in L_\infty(M)$ satisfying (3) and (4) <u>orthogonal sign functions</u>.

Laasonen [6] showed that a Tschebyscheff system $\{u_1, \ldots, u_m\}$ on a finite interval I possesses a unique partition $x_1 < \ldots < x_m$ such that

$$\sigma_m(x) := \text{sign}\, \omega_m(x) := \text{sign} \prod_{i=1}^{m} (x - x_i)$$

is a minimal (with smallest number of sign changes) orthogonal sign function of $\text{span}\{u_1, \ldots, u_m\}$. Therefore one can achieve a unique L_1-proximum of a function $f \in C_1(I)$ by the Lagrange interpolant $L_m f$ at the points x_i, if $\text{sign}(f - L_m f) = \varepsilon\sigma_m$ on $I \setminus Z(f - L_m f)$, $\varepsilon \in \{-1,1\}$.

<u>2. Unique L_1-approximation by blending interpolation.</u> Given two ECT-systems $U_m \subset C_1^m(I)$, $V_n \subset C_1^n(J)$ we define the <u>blending grid</u> $G_{m,n}$ by

$$G_{m,n} := \{(x,y) \in G := I \times J \mid \sigma_m(x)\sigma_n(y) =: \sigma_{m,n}(x,y) = 0\}$$

and the <u>blending interpolation operator</u> by the Boolean sum (cf. Gordon [3])

$$L_{m,n} := L_m^x \oplus L_n^y := L_m^x + L_n^y - L_m^x \circ L_n^y \ ,$$

where L_m^x and L_n^y denote the parametric Lagrange interpolation operators at the zeros of σ_m and σ_n, respectively.

<u>Definition 1.</u> Let $f \in C_1(G)$ and $L_{m,n} f$ as above.

(i) f is said to be <u>weakly adjoined</u> to $B(U_m,V_n) := B^{o,o}(U_m,V_n)$, if

$$(6) \qquad \sigma_{m,n}(x,y) = \varepsilon\, \text{sign}(f(x,y) - L_{m,n} f(x,y))$$

for all $(x,y) \in \mathring{G} \setminus Z(f - L_{m,n} f)$, $\varepsilon \in \{-1,1\}$.

(ii) f is <u>strongly adjoined</u> to $B(U_m,V_n)$, if (6) holds true in $\mathring{G}$.

Theorem 1. If $f \in C_1(G)$ is weakly adjoined to $B(U_m,V_n)$, then the unique L_1-proximum of f in $B(U_m,V_n)$ is given by the blending interpolant $L_{m,n}f$. The approximation constant is

$$E^1_{B(U_m,V_n)}(f) = |\int_G f\sigma_{m,n}| .$$

Proof. The fact, that $L_{m,n}f$ is an L_1-proximum is obvious and the approximation constant follows from (4),(5) and (6). If f is strongly adjoined to $B(U_m,V_n)$, the uniqueness follows from the fact, that for another L_1-proximum b of f the inequality $(f - L_{m,n}f)(f - b) \geq 0$ holds true (cf. Rice [8]) and from the uniqueness of the blending interpolant on $G_{m,n}$. If f is weakly adjoined, we use a method by Haußmann-Zeller [4]:

$$g_\varepsilon(x,y) := f(x,y) + \varepsilon\omega_m(x)\omega_n(y) , \quad \varepsilon \text{ as in Definition 1 },$$

is strongly adjoined to $B(U_m,V_n)$ and

$$\|f - L_{m,n}f\|_1 + \|\omega_m\omega_n\|_1 = \|g_\varepsilon - L_{m,n}f\|_1 \leq \|g_\varepsilon - b\|_1$$
$$\leq \|f - b\|_1 + \|\omega_m\omega_n\|_1 = \|f - L_{m,n}f\|_1 + \|\omega_m\omega_n\|_1$$

for all L_1-proxima b of f. To complete the proof, we get $b = L_{m,n}f$ by the uniqueness of the L_1-proximum of g_ε. □

Let Δ^x_m and Δ^y_n be the differential operators defined by U_m and V_n, respectively. Then we use $\Delta_{m,n} := \Delta^x_m \circ \Delta^y_n$ with $\ker \Delta_{m,n} = B^{m,n}(U_m,V_n)$. Let $C_1^{m,n}(G) := \{f \in C^{m,n}(G) \mid f^{(i,j)} \in L_1(G),\ 0 \leq i \leq m,\ 0 \leq j \leq n\}$.

Theorem 2. Let $f \in C_1^{m,n}(G)$ and $\Delta_{m,n}f \geq 0$. Then f is weakly adjoined to $B^{m,n}(U_m,V_n)$ and we get the unique L_1-proximum by blending interpolation.

Proof. By taking $\alpha_{m+1} := 1$ we construct by (1) an adjoined function (in the normal sense of Tschebyscheff systems) $\phi_{m+1}(x,x_o)$ to U_m and with $\beta_{n+1} := 1$ the function $\psi_{n+1}(y,y_o)$ adjoined to V_n, analogously. Then there exist functions $u \in \operatorname{span}\{\phi_1,\ldots,\phi_{m+1}\}$, $v \in \operatorname{span}\{\psi_1,\ldots,\psi_{n+1}\}$ with $\operatorname{sign} u = \sigma_m$, $\operatorname{sign} v = \sigma_n$. Let $w(x,y) := u(x)v(y)$. For arbitrary $(x_o,y_o) \in \overset{o}{G} \setminus G_{m,n}$ we choose $k := k(x_o,y_o) \in \mathbb{R}$ such that $h := f - L_{m,n}f + kw$ vanishes in $G_{m,n} \cup \{(x_o,y_o)\}$. By the mean-value theorem by Pólya [7] we get $\eta \in I$ and then $\xi \in J$ such that $\Delta_{m,n}h(\eta,\xi) = 0$. Then, by construction, $\Delta_{m,n}w \neq 0$ and constant, and we obtain

$$f(x_o,y_o) - L_{m,n}f(x_o,y_o) = \frac{\Delta_{m,n}f(\eta,\xi)}{\Delta_{m,n}w(\eta,\xi)}\, w(x_o,y_o) ,$$

such that

$$\operatorname{sign}(f - L_{m,n}f) = \varepsilon \operatorname{sign} w = \varepsilon\sigma_{m,n}$$

on $\overset{o}{G} \setminus Z(f - L_{m,n}f)$ with $\varepsilon \in \{-1,1\}$. This is the weak adjoinedness of f to $B^{m,n}(U_m,V_n)$ and Theorem 1 completes the proof. □

3. Estimates for the L_1-approximation constants. The parametric Taylor operators T_m^x, T_n^y defined by (2) commute, such that we get for $f \in C_1^{m,n}(G)$ and $(x_o,y_o) \in G$ the blending Taylor series by the Boolean sum

$$f(x,y) = T_{m,n}(f,x_o,y_o)(x,y) + R_{m,n}(f,x_o,y_o)(x,y)$$

$$= T_m^x(f,x_o)(x,y) + T_n^y(f,y_o)(x,y) - T_m^x(T_n^y(f,y_o),x_o)(x,y) + R_m^x(R_n^y(f,y_o),x_o)(x,y) ,$$

where $T_{m,n}(f,x_o,y_o) \in B^{m,n}(U_m,V_n)$. Using the Tschebyscheff spline functions $(1 \le i \le m)$

$$\phi_i^+(x,x_o) := \begin{cases} \phi_i(x,x_o) & \text{for } x_o < x , \\ 0 & \text{for } x_o \ge x , \end{cases}$$

$$\phi_i^-(x,x_o) := \begin{cases} -\phi_i(x,x_o) & \text{for } x_o > x , \\ 0 & \text{for } x_o \le x , \end{cases}$$

we define the kernel

$$\Phi_m(x_o,x) := \begin{cases} \int_a^{x_o} \phi_m^-(s,x)\sigma_m(s)ds & \text{for } x \le x_o , \\ \int_{x_o}^{b} \phi_m^+(s,x)\sigma_m(s)ds & \text{for } x \ge x_o \end{cases}$$

with $a := \inf I$, $b := \sup I$ and $x_o \in I$. Using the adjoined differential operators $D_i^* := (1/\alpha_i(x))\frac{d}{dx}$ and $\Delta_i^* := D_i^* \circ D_{i+1}^* \circ \cdots \circ D_m^*$ one can see, that $\Delta_i^* \Phi_m(x_o,x)\,|_{x=a,b} = 0$ for $m \ge i \ge 2$, $\Phi_m(x_o,a) = \Phi_m(x_o,b) = 0$ and $\Phi_m(x_o,x) \neq 0$ for all $x \in \overset{o}{I}$.

In a similar way we define ψ_j^+, ψ_j^- $(1 \le j \le n)$ and $\Psi_n(y_o,y)$ with $y_o \in J$, $c := \inf J$ and $d := \sup J$. Let $\Omega_{m,n} := \Phi_m \cdot \Psi_n$.

Lemma 1. For $f \in C_1^{m,n}(G)$ and any $(x_o,y_o) \in G$ we have

$$\int_G f\sigma_{m,n} = \int_G \Delta_{m,n} f \; \Omega_{m,n}(x_o,y_o) .$$

Proof. By the blending Taylor series of f we get

$$\int_G f\sigma_{m,n} = \int_G R_{m,n}(f,x_o,y_o)\sigma_{m,n}$$

$$= \int_I \int_J \int_{x_o}^{x} \int_{y_o}^{y} \Delta_{m,n} f(s,t)\phi_m(x,s)\psi_n(y,t)dtds\; \sigma_{m,n}(x,y)dydx .$$

Using the above definitions an easy calculation yields the result. □

Corollary 1. Let $f \in C_1^{m,n}(G)$. Then for any $(x_o,y_o) \in G$ we have

$$E^1_{B^{m,n}(U_m,V_n)}(f) \ge \left|\int_G \Delta_{m,n} f \; \Omega_{m,n}(x_o,y_o)\right|$$

with equality if $\Delta_{m,n} f \ge 0$. □

There is a monotonicity in the L_1-approximation constants:

Lemma 2. Let $f,g \in C_1^{m,n}(G)$ and $|\Delta_{m,n} f| \le \Delta_{m,n} g$. Then

$$E^1_{B^{m,n}(U_m,V_n)}(f) \le E^1_{B^{m,n}(U_m,V_n)}(g) \ .$$

Proof. We have $\Delta_{m,n} g \ge 0$ and $\Delta_{m,n}(g \pm f) \ge 0$, such that g and $g \pm f$ fulfill the assumptions of Corollary 1 and we get for any $(x_o,y_o) \in G$

$$E^1_{B^{m,n}(U_m,V_n)}(g) = \varepsilon \int_G g\sigma_{m,n} = \varepsilon \int_G \Delta_{m,n} g \ \Omega_{m,n}(x_o,y_o) \ ,$$

$$E^1_{B^{m,n}(U_m,V_n)}(g \pm f) = \varepsilon \int_G (g \pm f)\sigma_{m,n} = \varepsilon \int_G \Delta_{m,n}(g \pm f) \ \Omega_{m,n}(x_o,y_o)$$

with the same $\varepsilon \in \{-1,1\}$ in each of these cases. From these inequalities we get (the L_1-proxima are the blending interpolants)

$$0 \le \varepsilon\sigma_{m,n}(g - L_{m,n} g) \ ,$$

$$0 \le \varepsilon\sigma_{m,n}(g - L_{m,n} g) \pm \varepsilon\sigma_{m,n}(f - L_{m,n} f) \ ,$$

such that $|f - L_{m,n} f| \le |g - L_{m,n} g|$ yields

$$E^1_{B^{m,n}(U_m,V_n)}(f) \le \| f - L_{m,n} f \|_1$$

$$\le \| g - L_{m,n} g \|_1 = E^1_{B^{m,n}(U_m,V_n)}(g) \ . \ \square$$

Corollary 2. Let $f \in C_1^{m,n}(G)$ and $(x_o,y_o) \in G$. Then

$$E^1_{B^{m,n}(U_m,V_n)}(f) \le \left| \int_G |\Delta_{m,n} f| \Omega_{m,n}(x_o,y_o) \right| \ .$$

Proof. Let $g(x,y) := \int_{x_o}^{x} \int_{y_o}^{y} |\Delta_{m,n} f(s,t)| \phi_m(x,s) \psi_n(y,t) dtds$, then $\Delta_{m,n} g = |\Delta_{m,n} f|$. Lemma 2 and Corollary 1 yield the desired result. $\square$

As $\Omega_{m,n}(x_o,y_o) \in C(\overline{G}) \subset L_q(G)$ for $1 \le q \le \infty$, we get

Corollary 3. Let $f \in C_1^{m,n}(G)$ and $\Delta_{m,n} f \in L_p(G)$, $1 \le p \le \infty$, then we have for any $(x_o,y_o) \in G$

$$E^1_{B^{m,n}(U_m,V_n)}(f) \le \| \Delta_{m,n} f \|_p \| \Omega_{m,n}(x_o,y_o) \|_q$$

$$= \| \Delta_{m,n} f \|_p \| \Phi_m(x_o) \|_q \| \Psi_n(y_o) \|_q$$

with $1/p + 1/q = 1$ for $p \ne 1,\infty$, $q = \infty$ for $p = 1$ and $q = 1$ for $p = \infty$. $\square$

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