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ON THE APPROXIMATION OF HOLOMORPHIC FUNCTIONS BY MÜNTZ POLYNOMIALS ON AN INTERVAL AWAY FROM THE ORIGIN

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1. *Introduction.* Let C[a,b] denote the space of real valued continuous functions on [a,b] endowed with the uniform norm

$$\|f\|_{[a,b]} := \max\{|f(x)|: x\in[a,b]\}, \qquad f\in C[a,b].$$

Let (λ_ν) be a given sequence of real positive numbers λ_ν, $\nu\in\mathbb{N}$,

$$0\le\lambda_0<\lambda_1<\dots, \qquad \lim_{\nu\to\infty}\lambda_\nu=\infty. \tag{1}$$

With such a sequence (λ_ν) and a number $n\in\mathbb{N}$ let $\Pi_n(\lambda_\nu)$ denote the space of Müntz polynomials

$$\Pi_n(\lambda_\nu) := \{\sum_{\nu=0}^{n} a_\nu x^{\lambda_\nu}: a_\nu\in\mathbb{R}\}.$$

Considering the problem of the approximation of functions $f\in C[a,b]$, $0\le a<b<\infty$, by Müntz polynomials from $\Pi_n(\lambda_\nu)$ on [a,b] we set

$$\rho_n(f,(\lambda_\nu),[a,b]) := \min\{\|f-p_n\|_{[a,b]}: p_n\in\Pi_n(\lambda_\nu)\}.$$

For the rate of convergence of $\rho_n(f,(\lambda_\nu),[a,b])$ as $n\to\infty$ " Jackson typ theorems " have been proved by several authors (cf. e.g. [1],[2]).

In this paper we are interested in the question of what kind of functions $f\in C[a,b]$ can be approximated by Müntz polynomials from $\Pi_n(\lambda_\nu)$ particularly well in the sense that the minimal deviation tends to zero with a geometric rate , i.e.

$$\rho_n(f,(\lambda_\nu),[a,b]) \le A\cdot q^{-n} \qquad \text{for } n\in\mathbb{N} \text{ with } q>1.$$

For the approximation on the interval [0,1] this problem has been regarded in [5]. The results suggest that under the assumption $0<d\le\lambda_{\nu+1}-\lambda_\nu$ on the sequence (λ_ν) the minimal deviation $\rho_n(f,(\lambda_\nu),[0,1])$ tends to zero geometricly iff the approximated function f is the

restriction of a "Müntz series"

$$\hat{f}(z) = \sum_{\nu=o}^{\infty} c_\nu z^{\lambda_\nu} \quad , \; c_\nu \in \mathbb{R} \quad , \; z \in \mathbb{C}_{\log},$$

absolutely convergent in a certain domain around the branch point zero of the Riemann surface of the logarithm. (Since λ_ν is allowed to be irrational the complex value z must be an element of the Riemann surface of the logarithm denoted by $\mathbb{C}_{\log}$.)

2. The approximation of holomorphic functions on an interval away from the origin. We now ask what kind of functions can be approximated by Müntz polynomials with a geometric rate of convergence on an interval [a,b] where a>o.

M. Hasson [3] has regarded the special "Müntz problem" of the approximation of functions by usual polynomials p_n in which for a fixed $k \in \mathbb{N}$ the coefficients of x^k are zero, i.e.

$$p_n(x) = \sum_{\substack{\nu=o \\ \nu \neq k}}^{n} a_\nu x^\nu \; .$$

We define

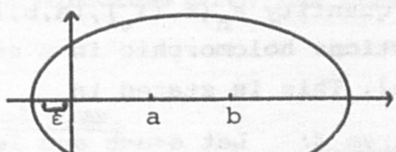
$$\rho_n^{(k)}(f,(\nu),[a,b]) := \min_{\alpha_\nu} \{ \| f - \sum_{\substack{\nu=o \\ \nu \neq k}}^{n} \alpha_\nu x^\nu \|_{[a,b]} \}$$

for $f \in C[a,b]$; $n,k \in \mathbb{N}$; k fixed. Since

$$\rho_n^{(k)}(x^k,(\nu),[o,1]) = O\left(\frac{1}{n^{2k}}\right) \qquad \text{as } n \to \infty$$

(cf. [3]) the minimal deviation on the interval [o,1] doesn't tend to zero geometricly even if the holomorphic function $f(x)=x^k$ is approximated by the other powers x^ν, $\nu \in \mathbb{N}$, $\nu \neq k$. The situation is changed if the interval is moved away from the origin. Indeed for the approximation on intervals [a,b] with a>O we have (cf. [3]

Theorem 1: Let a,b; o<a<b and $k \in \mathbb{N}$ be given. Let the function f be holomorphic in the interior of the ellipse $E(\frac{b+a}{2}+\varepsilon)$ with foci a,b and great half axe of length $\frac{b+a}{2}+\varepsilon$ with a number $\varepsilon>o$. In addition, suppose f(z) is real for real z and $f^{(k)}(o) \neq o$. Then there exist constants $c_1, c_2 > o$ such that inequality



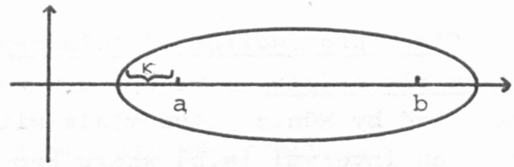
$$c_1 n^{-k} q^{-n} \leq \rho_n^{(k)}(f,(\nu),[a,b]) \leq c_2 n^{-k} q^{-n} \qquad (2)$$

holds for $n \in \mathbb{N}$ with q,

$$q := \frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}} \; .$$

A geometric convergence of $\rho_n^{(k)}(f,(\nu),[a,b])$ also occurs with a smaller rate for functions holomorphic in a smaller region around the approximation interval $[a,b]$, $a>o$. Combining the statement of Theorem 1 the inequality of W. A. Markoff (cf. [4]) and the inequality of S. N. Bernstein (cf. [4], p.92) we can prove

Theorem 2: Let $o<a<b$ and $k\in\mathbb{N}$ be given. Let $E(\frac{b-a}{2}+\kappa)$ denote the ellipse with foci a,b and great half axe of length $\frac{b-a}{2}+\kappa$ with a number $o<\kappa\leq a$. Suppose the function f is holomorphic in a region containing the ellipse $E(\frac{b-a}{2}+\kappa)$ and its interior. Further let f(z) be real for real z. Then with a constant A we have for $n\in\mathbb{N}$



$$\rho_n^{(k)}(f,(\nu),[a,b]) \leq A\, n^k \left(\frac{\sqrt{b+\kappa}+\sqrt{a-\kappa}}{\sqrt{b+\kappa}-\sqrt{a-\kappa}}\right)^n \left(\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}\right)^{-n} . \qquad (3)$$

Remark: For fixed a,b the function (cf. (3))

$$H(\kappa) := \frac{\sqrt{b+\kappa}+\sqrt{a-\kappa}}{\sqrt{b+\kappa}-\sqrt{a-\kappa}}$$

is a strictly monotonically decreasing function of κ, $o\leq\kappa\leq a$, with $H(o)=\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}$ and $H(a)=1$. For the special case $\kappa=a$ Theorem 2 includes (exept for the factor n^{2k}) the right inequality of (2) in Theorem 1.

Theorem 2 shows that for the special approximation problem above the minimal deviation $\rho_n^{(k)}(f,(\nu),[a,b])$ on an interval $[a,b]$, $a>o$ decreases geometricly to zero as $n\to\infty$ for all functions f holomorphic around the interval [a,b]. But also in the general case of Müntz approximation on [a,b] with a>o (in contrast to the approximation on [o,b]) the quantity $\rho_n(f,(\lambda_\nu),[a,b])$ has a geometric convergence for all functions holomorphic in a certain region containing the interval [a,b]. This is stated in

Theorem 3: Let $o<a<b$ and let (λ_ν) be a given sequence (1) satisfying $o<d\leq\lambda_{\nu+1}-\lambda_\nu\leq D<\infty$, $\nu\in\mathbb{N}$. Suppose the function f is holomorphic in the interior of K_R and continuous on K_R, $K_R:=\{z\in\mathbb{C}: |z|\leq R\}$ with $R>b$. Further, suppose f(z) is real for real z. Then for any q,

$$q<\sigma:=\min\left\{\left(\frac{R}{b}\right)^d, \frac{\sqrt{b^D}+\sqrt{a^D}}{\sqrt{b^D}-\sqrt{a^D}}\right\} \qquad (4)$$

there exists a constant A(q) such that for $n\in\mathbb{N}$

$$\rho_n(f,(\lambda_\nu),[a,b])\leq A(q)\,q^{-n} . \qquad (5)$$

Remark: Applying Theorem 2, under certain assumptions on the sequence (λ_ν) we also get a geometric decrease of $\rho_n(f,(\lambda_\nu),[a,b])$, $a>o$, for functions holomorphic only in smaller ellipses around $[a,b]$.

The upper bound (5) for $\rho_n(f,(\lambda_\nu),[a,b])$ depends on the value σ (cf. (4)). For functions f holomorphic in a sufficiently large region, i.e.

$$\left(\frac{R}{b}\right)^d \geq \frac{\sqrt{b^D}+\sqrt{a^D}}{\sqrt{b^D}-\sqrt{a^D}} \qquad (6)$$

the geometric rate of the minimal deviation is bounded by the right side of (6) no more depending on f. In this case the rate of convergence increases with the distance of the approximation interval $[a,b]$ from the origin. But generally for fixed interval $[a,b]$ even for the class of functions holomorphic in the whole plane the rate of $\rho_n(f,(\lambda_\nu),[a,b])$ is seen to be bounded by a constant depending on the distance of $[a,b]$ from the zero point. In fact we can prove

Theorem 4: Let a,b ; $o<a<b$ be given and let (λ_ν) be a sequence (1) satisfying $o<d\leq\lambda_{\nu+1}-\lambda_\nu\leq D<\infty$, $\nu\in\mathbb{N}$. Suppose for a function $f\in C[a,b]$ inequality

$$\rho_n(f,(\lambda_\nu),[a,b]) \leq A\rho^{-n}, \qquad n\in\mathbb{N},$$

holds with a constant A and ρ,

$$\rho > \kappa := \left(\frac{8\,b^D e}{a^D}\right)\left(\frac{\sqrt{b^d}+\sqrt{a^d}}{\sqrt{b^d}-\sqrt{a^d}}\right).$$

Then there exists a Müntz series

$$\hat{f}(z) = \sum_{\nu=o}^{\infty} c_\nu z^{\lambda_\nu}$$

absolutely convergent in $\overset{o}{K}_R := \{z\in\mathbb{C}_{log}: |z|<R\}$ with $R=\frac{\rho}{\kappa}$ whose restriction to the interval $[a,b]$ coincides with the given function f.

3. *Linear approximation with exponential sums.*

With aid of the transformation

$$x=e^{-t}, \qquad t\in[-\log b,-\log a], \qquad F(t)=f(e^{-t})$$
$$\qquad (7)$$
$$t=-\log x, \qquad x\in[a,b], \qquad f(x)=F(-\log x)$$

we see that the problem of the approximation of a function $f(x)\in C[a,b]$, $o\leq a<b<\infty$, by Müntz polynomials from $\Pi_n(\lambda_\nu)$ on $[a,b]$ is equivalent to the problem of the approximation of a function $F(t)\in C[\alpha,\beta]$ with $\alpha=-\log b$, $\beta=-\log a$ on $[\alpha,\beta]$ by help of exponential sums from $\Delta_n(\lambda_\nu)$,

$$\Delta_n(\lambda_\nu) := \{\sum_{\nu=0}^{n} a_\nu e^{-\lambda_\nu t} : a_\nu \in \mathbb{R}\}$$

where the minimal deviation is defined by

$$\delta_n(F,(\lambda_\nu),[\alpha\ \beta]) = \min\{\|F-d_n\|_{[\alpha\ \beta]}: d_n \in \Delta_n(\lambda_\nu)\}.$$

We now summarize the results stated above in terms of the (linear) approximation by exponential sums.

Noticing that under the transformation (7) the interval $[o,b]$, $b<\infty$, corresponds to the infinite interval $[\alpha,\infty]$ with $\alpha=-\log b$ the results in [5] suggest that a geometric rate for the minimal deviation $\delta_n(F,(\lambda_\nu),[\alpha,\infty])$ in approximating a function F by sums from $\Delta_n(\lambda_\nu)$ where the fixed sequence (λ_ν) satisfies $o<d\leq\lambda_{\nu+1}-\lambda_\nu$, $\nu\in\mathbb{N}$, occurs exactly for those functions F which are restrictions of Dirichlet series

$$\hat{F}(s) = \sum_{\nu=0}^{\infty} c_\nu e^{-\lambda_\nu s}, \qquad s=t+i\tau\in\mathbb{C},$$

convergent in certain right half planes. Whereas for the approximation on a finite interval $[\alpha,\beta]$ (corresponding to $[a,b]$, $o<a=e^{-\beta}<b=e^{-\alpha}<\infty$) the minimal deviation $\delta_n(F,(\lambda_\nu),[\alpha,\beta])$ for the exponential approximation where the numbers λ_ν, $\nu\in\mathbb{N}$, satisfy $\lambda_{\nu+1}-\lambda_\nu\leq D<\infty$ tends to zero geometricly for all functions F holomorphic in certain regions **around** the interval $[\alpha,\beta]$.

The difference between the approximation by exponential sums on $[\alpha,\infty]$ and the approximation on a finite interval $[\alpha,\beta]$, $\beta<\infty$, will be demonstrated by a

Numerical example:

We consider the approximation of the function $F(t)=\frac{1}{1+t}$ by exponential sums of the form (i.e. $\lambda_\nu=\frac{\nu}{2}$, $\nu\in\mathbb{N}$)

$$d_n(t) = \sum_{\nu=0}^{n} a_\nu e^{-\frac{\nu}{2}t}$$

on the interval $[o,\infty]$ resp. $[o,1]$. (F is holomorphic in the half plane $\operatorname{re} s>-1$ where $s=t+i\tau\in\mathbb{C}$, but not representable as a Dirichlet series)

The computed minimal deviations $\delta_n([o,\infty]) := \delta_n(F,(\frac{\nu}{2}),[o,\infty])$ resp. $\delta_n([o,1]) := \delta_n(F,(\frac{\nu}{2}),[o,1])$ are listed in Table 1 below for n=2,...,6. The last column of Table 1 gives as comparison the minimal deviations $\rho_n([o,1]) := \rho_n(F,(\nu),[o,1])$ for the approximation of $F(x)=\frac{1}{1}$ by algebraic polynomials on the same interval $[o,1]$.

Table 1:

n	$\delta_n([o,\infty])$	$\delta_n([o,1])$	$\rho_n([o,1])$
2	$4.5\cdot 1o^{-2}$	$3.3\cdot 1o^{-3}$	$7.3\cdot 1o^{-3}$
3	$3.4\cdot 1o^{-2}$	$3.9\cdot 1o^{-4}$	$1.3\cdot 1o^{-3}$
4	$3.1\cdot 1o^{-2}$	$4.7\cdot 1o^{-5}$	$2.2\cdot 1o^{-4}$
5	$2.8\cdot 1o^{-2}$	$5.4\cdot 1o^{-6}$	$3.7\cdot 1o^{-5}$
6	$2.6\cdot 1o^{-2}$	$6.6\cdot 1o^{-7}$	$6.4\cdot 1o^{-6}$

Table 2 contains the ratios of consecutive minimal deviations.

Table 2:

n	$\frac{\delta_n([o,\infty])}{\delta_{n+1}([o,\infty])}$	$\frac{\delta_n([o,1])}{\delta_{n+1}([o,1])}$	$\frac{\rho_n([o,1])}{\rho_{n+1}([o,1])}$
2	1.3o	8.46	5.62
3	1.11	8.3o	5.91
4	1.1o	8.28	5.95
5	1.o8	8.18	5.96

The ratios for the exponential approximation on $[o,\infty]$ (first column, Table 2) indicate that a geometric convergence of $\delta_n([o,\infty])$ cannot be expected. For the approximation on $[o,1]$ a geometric rate occurs in the exponential case as in the polynomial case.
In fact the theorems of Bernstein (cf. [4]) lead to the asymptotic relation

$$\rho_n([o,1]) = O\left((3+\sqrt{1o})^{-n} \right) \quad \text{where } 3+\sqrt{1o} \approx 6.16$$

for $n\to\infty$ and with some transformation arguments we can find

$$\delta_n([o,1]) = O\left(\kappa^{-n} \right) \quad \text{with } \kappa := \frac{1+\sqrt{e}+\sqrt{e+2\sqrt{e}}}{\sqrt{e}-1} \approx 7.86 .$$

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