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ON THE BANACH SPACE OF UNIFORMLY STRONG CONVERGENT FOURIER SERIES

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The concept of strong summability may be said to have originated from a paper of Hardy and Littlewood in 1913 in connection with Fourier series (see [4] p. 59). The first definition was given by Fekete [1] in 1916, that is the series

(1) $$\sum_{n=0}^{\infty} c_n$$

is strongly summable to the number s if $\sum_{k=0}^{n} |s_k - s| = o(n)$ $(n \to \infty)$ where s_k is the k-th partial sum of the series (1). In 1933 Winn [7] defined strong Cesaro summability for positive orders, namely the series (1) is said to be strongly summable (C,α) to the number s if $\sum_{k=0}^{n} |\sigma_k^{(\alpha-1)} - s| = o(n)$, where the parameter $\alpha > 0$,

(2) $$\sigma_k^{(\alpha-1)} = \frac{1}{A_k^{(\alpha-1)}} \sum_{\nu=0}^{k} A_{k-\nu}^{(\alpha-1)} c_\nu$$

and

(3) $$A_0^{(\alpha-1)} = 1, \qquad A_\nu^{(\alpha-1)} = \frac{\alpha(1+\alpha)(2+\alpha)\dots(\nu-1+\alpha)}{\nu!} \quad (\nu = 1,2,\dots).$$

The strong $(C,0)$ summability might have been strong convergence but by (3) we can see that (2) is cannot be extended to $\alpha = 0$, so a definition has to be found for the strong convergence which was suitable for consistency theorems of ordinary Cesaro, strong

Cesaro and absolute Cesaro summabilities.

The first definition corresponding to strong convergence was given in 1949 by Hyslop [2] as follows, the series (1) is said to be strongly convergent if it is convergent and if

$$\sum_{k=1}^{n} k|c_k| = o(n) \qquad (n \to \infty).$$

This definition shows that the strong convergence implies the ordinary convergence and by the transformation

$$\sum_{k=1}^{n} k|c_k| = \sum_{l=1}^{n} \sum_{k=l}^{n} |c_l| \leq \sum_{l=1}^{n} \sum_{k=l}^{\infty} |c_l|$$

we can see that the absolute convergence implies the strong convergence.

Recently Tanovic-Miller [6] introduced a new definition of strong convergence saying that the series (1) is strongly convergent to the number s if

$$\sum_{k=1}^{n} |(k+1)(s_k - s) - k(s_{k-1} - s)| = o(n) \qquad (n \to \infty)$$

where $s_{-1} = 0$. Considering the identity

$$s_n - s = \frac{1}{n+1} \sum_{k=0}^{n} \left((k+1)(s_k - s) - k(s_{k-1} - s)\right) =$$

(4)

$$= \frac{1}{n+1} \sum_{k=0}^{n} k c_k + \frac{1}{n+1} \sum_{k=0}^{n} (s_k - s)$$

it is easy to see that Hyslop's definition and Tanovic-Miller's definition are equivalent.

Denote by C the Banach space of 2π periodical continuous functions with the norm

$$\|f\|_C = \sup_{0 \leq t \leq 2\pi} |f(t)|.$$

Using the usual notations $s_n(f,t)$ means the n-th partial sum of the Fourier series of f at the point t

$$\frac{a_0(f)}{2} + \sum_{n=1}^{\infty} (a_n(f) \cos nt + b_n(f) \sin nt) \equiv \sum_{n=0}^{\infty} A_n(f,t).$$

Denote by U, S and A the classes of functions f belonging to C whose Fourier series converges uniformly, uniformly strongly and absolutely on $[0,2\pi]$, respectively. These mean that $f \in \mathcal{U}$ if and only if

$$\lim_{n\to\infty} \|f - s_n(f)\|_C = 0 \quad , \tag{5}$$

$f \in \mathcal{S}$ if and only if (5) is fulfilled and

$$\lim_{n\to\infty} \frac{1}{n+1} \|\sum_{k=0}^{n} k\, |A_k(f)|\,\|_C = 0 \tag{6}$$

moreover $f \in \mathcal{A}$ if and only if

$$\frac{|a_0(f)|}{2} + \sum_{n=1}^{\infty} \left(|a_n(f)| + |b_n(f)|\right) < \infty .$$

Of course if a trigonometrical series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

converges uniformly, uniformly strongly or absolutely on $[0,2\pi]$, then its sum belongs to the class $\mathcal{U}$, $\mathcal{S}$ or $\mathcal{A}$, respectively.

It is known that $\mathcal{U}$ is a Banach space with the norm

$$\|f\|_{\mathcal{U}} = \sup_n \|s_n(f)\|_C \tag{7}$$

(see [3] p. 11-12) and $\mathcal{A}$ is a Banach space with the norm

$$\|f\|_{\mathcal{A}} = \frac{|a_0(f)|}{2} + \sum_{n=1}^{\infty} \left(|a_n(f)| + |b_n(f)|\right)$$

(see [3] p. 12).

Tanovic-Miller showed that $\mathcal{A}$ is a real subset of $\mathcal{S}$ which itself is a real subset of $\mathcal{U}$ ([6], Theorem 4). Our aim is to find a suitable norm for the $\mathcal{S}$ so that $\mathcal{S}$ should be a Banach space with this norm. By (4), (5) and (6) we have that for any $f \in \mathcal{S}$

$$\sup_n \frac{1}{n+1} \|\sum_{k=0}^{n} |(k+1)s_k(f) - k s_{k-1}(f)|\,\|_C \le$$

$$\le \sup_n \frac{1}{n+1} \|\sum_{k=1}^{n} k\,|A_k(f)|\,\|_C + \sup_n \frac{1}{n+1} \sum_{k=0}^{n} \|s_k(f) - f\|_C + \|f\|_C$$

and we can prove

<u>Theorem 1</u>. <u>The set $\mathcal{S}$ is a Banach space endowed with the norm</u>

$$\|f\|_{\mathcal{S}} = \sup_n \frac{1}{n+1} \|\sum_{k=0}^{n} |(k+1)s_k(f) - k s_{k-1}(f)|\,\|_C$$

<u>and for any</u> $f \in \mathcal{S}$ $\|f\|_{\mathcal{U}} \le \|f\|_{\mathcal{S}}$, <u>moreover for any</u> $f \in \mathcal{A}$ $\|f\|_{\mathcal{S}} \le 2\|f\|_{\mathcal{A}}$.

Of course we have to show that S is not a Banach space with the norm (7). Turning to this direction we mention that for any non negative integers n and m

$$\|s_n(f - s_m(f))\|_C = \begin{cases} 0 & \text{if} \quad n \leq m \\ \|s_n(f) - s_m(f)\|_C & \text{if} \quad n > m \end{cases}$$

holds. Assuming that $f \in U \setminus S$ we get

$$\lim_{m\to\infty} \|f - s_m(f)\|_U = \lim_{m\to\infty} \sup_{n>m} \|s_n(f) - s_m(f)\|_C = 0$$

and hence

$$\lim_{n,m\to\infty} \|s_n(f) - s_m(f)\|_U = 0$$

that is, $\{s_n(f)\}_{n=1}^{\infty}$ is a Cauchy sequence in S with the norm $\|\cdot\|_U$. As f does not belong to the class S, soo S is not a Banach space with the norm $\|\cdot\|_U$, moreover by Theorem 1 we have got

$$\sup_{\substack{f\in S\\ f\neq 0}} \frac{\|f\|_S}{\|f\|_U} = \infty .$$

If $f \in A$ then $\lim_{m\to\infty} \|f - s_m(f)\|_A = 0$ is obvious. A similar result is true for the class S. It is shown by

<u>Theorem 2</u>. <u>If</u> $f \in S$, <u>then</u> $\lim_{m\to\infty} \|f - s_m(f)\|_S = 0$. This theorem shows that the class A is not a Banach space with the norm $\|\cdot\|_S$ and so

$$\sup_{\substack{f\in A\\ f\neq 0}} \frac{\|f\|_A}{\|f\|_S} = \infty .$$

Finally we mention that S is not a Banach algebra. More emactly by the Banach-Steinhaus theorem we can prove

<u>Theorem 3</u>. <u>There exist such functions</u> $f \in A$ <u>and</u> $g \in S$ <u>that</u> $f \cdot g \notin U$. The full proofs of Theorems 1-3 are in [5].

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