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L^p-APPROXIMATION BY SPLINES

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1. Construction of the spline function. Let $[a,b]$ be an interval and $a=t_0<t_1<\dots<t_n=b$ an equidistant subdivision of $[a,b]$ with $h=t_{k+1}-t_k$. Let the y_k's be given numbers $(k=0,1,\dots,n)$ and we consider the function S defined by

$$S(t)=S_k(t) \tag{1}$$

for $t_k\le t\le t_{k+1}$, $k=0,1,\dots,n-1$, where

$$S_k(t)=y_k+\frac{1}{h}\Delta y_k(t-t_k)-\frac{1}{h^2}(\Delta y_{k+1}-\Delta y_k)\left[(t-t_k)^2-\frac{1}{h}(t-t_k)^3\right]. \tag{2}$$

Here we used the notations $y_{n+1}=0$, $\Delta y_k=y_{k+1}-y_k$. The function S is obviously continuously differentiable and $S(t_k)=y_k$ $(k=0,1,\dots,n)$.

If x is a continuously differentiable function and we let $y_k=x(t_k)$ $(k=0,1,\dots,n)$, then it is easy to prove (see [3]), that

$$\|x-S\|_\infty\le \text{const.}\, h\,\omega(h;x')$$

$$\|x'-S'\|_\infty\le \text{const.}\,\omega(h;x')$$

where $\omega(\cdot;x')$ denotes the modulus of continuity of the function x'. Further, in [1] it is proved, that if x is square-integrable with left and right limits at each point and we let $y_k=[x(t_k+0)+x(t_k-0)]\cdot\frac{1}{2}$ $(k=0,1,\dots,n)$, then

$$\|x-S\|_2\to 0$$

whenever $h\to 0$.

In this work our aim is to prove, that if x belongs to $L^p[a,b]$ with $1 \le p < +\infty$ and

(3) $$y_k = \frac{1}{h}\int_{t_k}^{t_k+h} x(t)\,dt \qquad (k=0,1,\dots,n)$$

then $\|x-S\|_p \to 0$ whenever $h \to 0$. Further, if x has an L^p-derivative x', then $\|x'-S'\|_p \to 0$ whenever $h \to 0$.
(Here, and in what follows, we always suppose, that x is extended in the following manner : $x(t)=0$ if $t<a$ or $t>b$.)
The proof is based on the following two facts: the L^p-modulus of continuity of any L^p-function converges to zero, and the Steklov-function of any L^p-function converges to the function in the L^p-norm.

2. Convergence of the spline function.

LEMMA Let $1 \le p < +\infty$ and $x \in L^p[a,b]$. Then we have

(i) $$\lim_{h\to 0}\int_a^b |x(t+h)-x(t)|^p\,dt = 0$$

(ii) $$\lim_{h\to 0}\int_a^b \Big| x(t) - \frac{1}{h}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} x(s)\,ds\Big|^p dt = 0 \quad .$$

For the proof of (i) see e.g. [2], and the proof of (ii) can be found in [4]. We shall use the notations

$$\omega_p(h;x) = \int_a^b |x(t+h)-x(t)|^p\,dt$$

and

$$\delta_p(h;x) = \int_a^b \Big| x(t) - \frac{1}{h}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} x(s)\,ds\Big|^p dt \quad .$$

The function $\omega_p(\cdot\,;x)$ is called the L^p-modulus of continuity of the function x, and the function

$$x_h(t) = \frac{1}{h}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} x(s)\,ds$$

is called the Steklov-function of x.

Our main theorem is the following:

THEOREM Let $1 \le p < +\infty$ and $x \in L^p[a,b]$. If S denotes the spline function defined by (1)-(3), then we have

$$\|x-S\|_p^p \le 8^{p-1}\delta_p(h;x) + \left(\frac{4^{p-1}}{p+1} + \frac{2\cdot 8^{p-1}}{2p+1} + \frac{2\cdot 8^{p-1}}{3p+1}\right)\omega_p(h;x) \,.$$

PROOF. In the sequel we shall make use of the elementary inequality

$$(a+b)^p \le 2^{p-1}\,(a^p+b^p)$$

which holds for $a, b \ge 0$, $p \ge 1$.

Then we can calculate as follows:

$$\|x-S\|_p^p = \int_a^b |x(t)-S(t)|^p\,dt =$$

$$= \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} \left| x(t) - y_k - \frac{1}{h}\Delta y_k (t-t_k) + \frac{1}{h^2}(\Delta y_{k+1} - \Delta y_k)(t-t_k)^2 - \frac{1}{h^3}(\Delta y_{k+1}-\Delta y_k)(t-t_k)^3\right|^p dt \le$$

$$\le 4^{p-1}\sum_{k=0}^{n-1}\left[\int_{t_k}^{t_{k+1}} |x(t)-y_k|^p\,dt + \frac{h}{p+1}|\Delta y_k|^p + \frac{h}{2p+1}|\Delta y_{k+1}-\Delta y_k|^p + \frac{h}{3p+1}|\Delta y_{k+1}-\Delta y_k|^p\right] \le$$

$$\le 4^{p-1}\sum_{k=0}^{n-1}\left[\int_{t_k}^{t_{k+1}} |x(t)-y_k|^p\,dt + \frac{h}{p+1}|\Delta y_k|^p + \frac{2^{p-1}h}{2p+1}\left(|\Delta y_{k+1}|^p + |\Delta y_k|^p\right) + \right.$$

$$\left. + \frac{2^{p-1}h}{3p+1}\left(|\Delta y_{k+1}|^p + |\Delta y_k|^p\right)\right].$$

The estimation of the first term is as follows:

$$\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} |x(t)-y_k|^p\,dt = \sum_{k=0}^{n-1}\int_{t_k}^{t_k+h}\left| x(t) - \frac{1}{h}\int_{t_k}^{t_k+h} x(s)\,ds\right|^p dt \le$$

$$\le 2^{p-1}\sum_{k=0}^{n-1}\left[\int_{t_k}^{t_k+h}\left| x(t) - \frac{1}{h}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} x(s)\,ds\right|^p dt + \int_{t_k}^{t_k+h}\left|\frac{1}{h}\int_{t-\frac{h}{2}}^{t+\frac{h}{2}} x(s)\,ds - \frac{1}{h}\int_{t_k}^{t_k+h} x(s)\,ds\right|^p dt\right] \le$$

$$\le 2^{p-1}\,\delta_p(h;x) + 2^{p-1}\sum_{k=0}^{n-1}\int_{t_k}^{t_k+h}\left|\frac{1}{h}\int_{t-\frac{h}{2}}^{t_k} x(s)\,ds - \frac{1}{h}\int_{t+\frac{h}{2}}^{t_k+h} x(s)\,ds\right|^p dt \le$$

$$\le 2^{p-1}\,\delta_p(h;x) + 2^{p-1}\sum_{k=0}^{n-1}\int_{t_k}^{t_k+h}\frac{1}{h^p}\int_{t_k}^{t_k+h}|x(s-h)-x(s)|^p\,ds\cdot h^{\frac{p}{q}}\,dt$$

by the Hölder-inequality, with $\frac{1}{p}+\frac{1}{q}=1$. But $\frac{p}{q}=p-1$, and hence we have

$$2^{p-1}\sum_{k=0}^{n-1}\int_{t_k}^{t_k+h}\frac{1}{h^p}\int_{t_k}^{t_k+h}|x(s-h)-x(s)|^p\,ds\cdot h^{p-1}\,dt =$$

$$= 2^{p-1}\sum_{k=0}^{n-1}\int_{t_k}^{t_k+h}|x(s-h)-x(s)|^p\,ds \le 2^{p-1}\,\omega_p(h;x).$$

The next step is to estimate expressions of the form

$$\sum_{k=0}^{n-1} h\,|\Delta y_k|^p .$$

We have

$$\sum_{k=0}^{n-1} h\,|\Delta y_k|^p = h \sum_{k=0}^{n-1} \left| \frac{1}{h}\int_{t_{k+1}}^{t_{k+1}+h} x(t)\,dt - \frac{1}{h}\int_{t_k}^{t_k+h} x(t)\,dt \right|^p \leq$$

$$\leq \sum_{k=0}^{n-1} \frac{1}{h^{p-1}} \left| \int_{t_k}^{t_k+h} |x(t+h)-x(t)|\,dt \right|^p \leq$$

$$\leq \sum_{k=0}^{n-1} \frac{1}{h^{p-1}} \int_{t_k}^{t_k+h} |x(t+h)-x(t)|^p\,dt \cdot h^{\frac{p}{q}} = \omega_p(h;x)$$

where we have used the Hölder-inequality again. Finally, substitution into the original inequality gives the statement.

3. Remarks. The above approximation process has the following stability property: if $x, \tilde{x} \in L^p[a,b]$, further S and $\tilde{S}$ denote the respective spline functions, then

$$\| x - \tilde{x} \|_p^p < \varepsilon$$

implies

$$\| S - \tilde{S} \|_p^p \leq 4^{p-1} \varepsilon \left(1 + \frac{2^p}{p+1} + \frac{2^{2p}}{2p+1} + \frac{2^{2p}}{3p+1} \right) .$$

The proof can be proceeded by an easy calculation.

Using the above methods we can prove the following theorem:

THEOREM Let $1 \leq p < +\infty$ and $x \in L^p[a,b]$. If x has an L^p-derivative x', and S denotes the spline function defined by (1)-(3), then we have

$$\| x' - S' \|_p \longrightarrow 0$$

whenever $h \to 0$.

The L^p-derivative x' of x is defined as the function $x' \in L^p[a,b]$ with the property, that

$$\lim_{h\to 0} \int_a^b \left| \frac{x(t+h)-x(t)}{h} - x'(t) \right|^p dt = 0 .$$

It is known (see [5]), that the function $x \in L^p[a,b]$ has an L^p-derivative if and only if x extended as above belongs to the Sobolev-space $W^{1,p}(\mathbb{R})$. Using the Sobolev-norm $\| \ \|_{1,p}$ in $W^{1,p}(\mathbb{R})$ we have:

COROLLARY Let $1 \le p < +\infty$ and $x \in W^{1,p}(\mathbb{R})$ identically zero outside the interval $[a,b]$. If S denotes the spline function defined by (1)-(3), then we have

$$\lim_{h \to 0} \| x - S \|_{1,p} = 0 .$$

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