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DIVIDED DIFFERENCES AND MODULI OF SMOOTHNESS OF FUNCTIONS, FUNCTION SUPERPOSITIONS AND THEIR APPLICATION

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1. Introduction. In the paper there are given identities for complex divided and finite differences of functions and their superpositions. On the basis of them estimates for moduli of smoothness of functions and their superpositions are obtained. The paper contains also some other results on properties of divided and finite differences and moduli of smoothness, on relations between different moduli of smoothness. Some applications are considered. The given results are a part of results of the preprint [1] which were not included into [2].

2. Problems on function superpositions. Investigating a problem concerning finite-difference smoothnesses of conformal mappings, E.W. Karupu has come to posing the problem of finite-difference smoothnesses of any natural order k for superpositions $f \circ g$ of functions g and f in connection with analogous smoothnesses of the functions f and g. A particular case of this problem for k = 2 and classical (arithmetical - in terminology of [3]) moduli of smoothness for concrete functions of real variables (concerning conformal mapping of a disk onto a smooth domain under a Hölder condition) was formerly treated by S.E. Warschawski [4] (in other formulation), R.N. Koval'čuk [5] and L.I. Kolesnik [6], and there it was solved by means of the method due to S.E. Warschawski [4] based on the introduction of additional points. This method was generalized by E.W. Karupu [7] for the case k > 2. But the mentioned method contains a roughening step: the replacement of finite differences (and

moduli of smoothness) of order k by finite differences (and moduli of smoothness) of order 2, and as a result non-sharp inequalities obtained by means of it do not possess some property important for applications, and have essentially a restricted range of applications.

The mentioned problem on finite-difference smoothnesses of function superpositions has been solved in the preprint [1] (see also [2]): there are given estimates for moduli of smoothness of the superposition $f\circ g$ via moduli of smoothness of f and g (the direct estimates) and estimates for moduli of smoothness of the function f via moduli of smoothness of $f\circ g$ and g (the inverse estimates). The main difficulties of derivation of these estimates are concentrated in the establishment of proper finite-difference identities giving an expression of the complex divided differences of the superposition $f\circ g$ via complex divided differences of f and g and divided differences of f via divided differences of $f\circ g$ and g. A property of these identities and estimates important for applications is named *ordinal homogeneity*. The results are obtained by a method not involving any additional points and attendent roughenings.

<u>3. Identities for divided differences of function superpositions.</u> A point collection $w_o,\dots,w_m$ in the complex plane C will be called *simple* if all the points $w_o,\dots,w_m$ mutually do not coincide. For a function h given on a simple point collection w_o, ..., w_m let $[w_o,\dots,w_m]_h$ and $[w_o,\dots,w_m;h,w_o]$ denote respectively the divided and the finite differences (see [3]).

Let m be a positive integer, j_o, $j_1,\dots,j_m$ be integers, and $j_o<j_1<\dots<j_m$. Let $z_{j_o},z_{j_1},\dots,z_{j_m}$ be a given simple point collection and g be a finite function given on it. For integers s, r, q let us introduce quantities $\alpha_{j_r,j_q}(z_{j_o},z_{j_1},\dots,z_{j_m})_g=\alpha_{j_r,j_q}$ by the formulae: if $0\leqslant r\leqslant q\leqslant m-1$ then

$$\alpha_{j_r,j_q}(z_{j_o},z_{j_1},\dots,z_{j_m})_g=\alpha_{j_r,j_q}$$

$$=\sum_{\substack{p(q+1),\dots,p(m-1)\\ r=p(q)\leqslant p(q+1)\leqslant\dots\leqslant p(m)=q}}\ \prod_{n=q}^{m-1}[z_{j_{p(n)}},z_{j_{p(n)+1}},\dots,z_{j_{p(n+1)}},z_{j_{n+1}}]_g, \qquad (1)$$

and if $0\leqslant q<r\leqslant m-1$ then

$$\alpha_{j_r,j_q}(z_{j_o},z_{j_1},\ldots,z_{j_m})_g = \alpha_{j_r,j_q} = 0. \tag{2}$$

Here and throughout this paper for $p(n+1) = p(n)$ the ordered collection of the shape $p(n),p(n)+1,\ldots,p(n+1)$ is understood as the one-element collection $p(n)$. In similar way there are interpreted the index collection of the shape $j_{p(n)},\ldots,j_{p(n+1)}$, the point collection of the shape $z_{p(n)},z_{p(n)+1},\ldots,z_{p(n+1)}$ and so on. For instance for $p(n+1) = p(n)$ the quantity $[z_{j_{p(n)}},z_{j_{p(n)+1}},\ldots,z_{j_{p(n+1)}},z_{j_{n+1}}]_g$ is defined as $[z_{j_{p(n)}},z_{j_{n+1}}]_g$.

From (1), (2) one can see that $\alpha_{j_r,j_q}(z_{j_o},z_{j_1},\ldots,z_{j_m})_g$ do not depend on the points z_{j_s} with indexes $s<r$ and on the values of g in them, and

$$\alpha_{j_r,j_q}(z_{j_o},z_{j_1},\ldots,z_{j_m})_g = \alpha_{j_r,j_q}(z_{j_r},z_{j_{r+1}},\ldots,z_{j_m})_g.$$

Let $\delta_{r,s}$ be the Kronecker symbol

$$\delta_{r,s} = \begin{cases} 1 & \text{if } r=s, \\ 0 & \text{if } r\neq s. \end{cases} \tag{3}$$

For $r,s = 0,\ldots,m-1$ let us denote

$$\alpha^{(s)}_{j_r,j_q}(z_{j_o},z_{j_1},\ldots,z_{j_m})_g = \begin{cases} \delta_{r,s} & \text{if } q=0, \\ \alpha_{j_r,j_q} & \text{if } q=1,\ldots,m-1. \end{cases} \tag{4}$$

Let us also denote

$$A_{j_s}(z_{j_o},z_{j_1},\ldots,z_{j_m})_g = \det \left\| \alpha^{(s)}_{j_r,j_q} \right\|_{r,q=0,\ldots,m-1} \quad (s=0,\ldots,m-1). \tag{5}$$

For points $w_o,w_1,\ldots,w_m \in C$ let $W(w_o,w_1,\ldots,w_m)$ be the Vandermonde determinant

$$W(w_o,\ldots,w_m) = \prod_{\substack{i,j \\ 0\leq i<j\leq m}} (w_i - w_j).$$

In these notations with a point collection $z_o,z_1,\ldots,z_k$ (k is a positive integer) as the point collection $z_{j_o},z_{j_1},\ldots,z_{j_m}$ the following results are true.

THEOREM 1. Let $z_o,\dots,z_k$ and $g_o,\dots,g_k$ be simple point collections on C, and a function g be given in $z_o,\dots,z_k$ by the equation $g(z_j) = g_j$, $j = 0,\dots,k$. Then for arbitrary function f defined in $g_o,\dots,g_k$ there hold the identities

$$[z_o,\dots,z_k]_{f\circ g} = \sum_{s=0}^{k-1} [g_s,\dots,g_k]_f \; \alpha_{o,s}(z_o,\dots,z_k)_g, \tag{6}$$

$$[z_o,\dots,z_k;f\circ g,z_o] - [g_o,\dots,g_k;f,g_o] = \Big(\prod_{r=1}^{k}(z_o - z_k)\Big)$$

$$\times \sum_{s=1}^{k-1} [g_s,\dots,g_k]_f \; \alpha_{o,s}(z_o,\dots,z_k)_g. \tag{6'}$$

THEOREM 2. Under the conditions of theorem 1 there hold the identities

$$[g_o,\dots,g_k]_f \frac{W(g_o,\dots,g_k)}{W(z_o,\dots,z_k)} = \sum_{s=0}^{k-1} [z_s,\dots,z_k]_{f\circ g} \; A_s(z_o,\dots,z_k)_g, \tag{7}$$

$$[g_o,\dots,g_k;f,g_o] - [z_o,\dots,z_k;f\circ g,z_o]$$

$$= \Big(\prod_{r=1}^{k}(z_o - z_r)\Big) \frac{W(z_1,\dots,z_k)}{W(g_1,\dots,g_k)} \sum_{s=1}^{k-1} [z_s,\dots,z_k]_{f\circ g} \; A_s(z_o,\dots,z_k)_g. \tag{7'}$$

In order to prove these theorems one must know the formulae (1) - (7′) explicitely. But they are very complicated, and discovering them was extremely arduous task for intuition and imagination. The theorems 1 and 2 are proved by induction in k = 1,2,... . Not giving the proofs we note that they make use of the following lemma which is proved by induction in m = 2,3,...,k.

LEMMA. For $m \geq 2$ there hold the equations

$$\alpha_{o,o}(z_o,z_2,\dots,z_m)_g - \alpha_{1,1}(z_1,z_2,\dots,z_m)_g$$

$$= (z_o - z_1)\,\alpha_{o,1}(z_o,z_1,\dots,z_m)_g,$$

$$\alpha_{o,q}(z_o,z_2,\dots,z_m)_g - \alpha_{1,q}(z_1,z_2,\dots,z_m)_g$$

$$= (z_o - z_1)\,\alpha_{o,q}(z_o,z_1,\dots,z_m)_g \qquad \forall q = 2,\dots,m-1.$$

Any divided difference of the shape

$$\beta = [z_{j_{p(n)}}, z_{j_{p(n)+1}}, \ldots, z_{j_{p(n+1)}}]_g$$

contained in (1) has the order $\gamma(\beta) = p(n+1) - p(n) + 1$ (see [3], p. 31). In accordance with the theorem 5.3.1 of [3] this divided difference can be expressed as a homogeneous linear combination of divided differences of the same order at point collections with neighbouring lower indexes:

$$\beta = \sum_{p=p(n)}^{n+1-\gamma(\beta)} [z_{j_p}, z_{j_{p+1}}, \ldots, z_{j_{p+\gamma(\beta)}}]_g \; a_p(\beta)$$

where each $a_p(\beta)$ does not depend on f and is a homogeneous polynomial of the order $\gamma(\beta)$ in partial fractions of the shape

$$(z_{j_\mu} - z_{j_\lambda})/(z_{j_h} - z_{j_t}) \tag{8}$$

with the indexes μ, λ, h, t such that $p(n) \leq h \leq \mu < \lambda \leq t \leq n+1$, all coefficients of this polynomial being only -1, 0, 1.

Let β_ν be a divided difference of a function $\xi_\nu(z)$ of the variable z, index ν running over a finite set of values. The o r d e r $\gamma(P)$ of a finite product $P = \prod_\nu \beta_\nu$ of divided differences β_ν is defined as the sum of orders of factors: $\gamma(P) = \sum_\nu \gamma(\beta_\nu)$ (here some of factors β_ν particularly may coincide).

If for u = 1,2,...,M each P_u is a finite product of divided differences and the order $\gamma(P_u) = \gamma$ does not depend on u, then to any homogeneous linear combination $L = \sum_u a_u P_u$ with coefficients a_u not depending on $z_o, \ldots, z_k$ we assign the order $\gamma(L) = \gamma$.

Then for $0 \leq r \leq q \leq m-1$ the expression (1) is the linear homogeneous combination $L_{j_r, j_q} = \sum_u a_u P_u$ of products $P_u = \prod_{v=1}^{m-q} \beta_{u,v}$ of divided differences of the form $\beta_{u,v} = [z_{j_\mu}, z_{j_{\mu+1}}, \ldots, z_{j_\lambda}]_g$, $r \leq \mu < \lambda \leq m$, where $\gamma(L_{j_r,j_q}) = m - r$ and a_u are homogeneous polynomials of the order $\gamma(L_{j_r,j_q}) = m - r$ in partial fractions of the form (8) with indexes μ, λ, h, t satisfying the conditions $r \leq h \leq \mu < \lambda \leq t \leq m$. All coefficients of the mentioned homogeneous polynomials are integers and are bounded by a quantity depending only on m.

Taking this into account, on the basis of (1) - (5) we conclude that the expression (5) is a linear combination $L = \sum_l b_l H_l$ of products $H_l = \prod_{v=1}^{m(m-1)/2} \beta_{(l,v)}$ of divided differences of the form

$$\beta_{(l,v)} = [z_{j_\mu}, z_{j_{\mu+1}}, \ldots, z_{j_\lambda}]_g \qquad (0 \leq \mu < \lambda \leq m),$$

where the order of each H_l is equal to

$$\sum_{r=0}^{m-1} (m - r) - (m - s) = \frac{m(m-1)}{2} + s \stackrel{\text{def}}{=} \gamma(m,s),$$

and each b_l is a homogeneous polynomial of the order $\gamma(m,s)$ in partial fractions of the shape (8) with indexes μ, λ, h, t satisfying the conditions $0 \leq h \leq \mu < \lambda \leq t \leq m$, and all coefficients of this polynomial are integers and are bounded by a quantity depending only on m.

Applying this to points $z_0, \ldots, z_k$ we obtain the following statement.

THEOREM 3. Under the conditions of theorems 1 - 2 the coefficients have the following representations:

$$\alpha_{r,s}(z_0,\ldots,z_k)_g = \sum_{\substack{u,l \\ u=(u_1,\ldots,u_{k-s}) \\ l=(l_1,\ldots,l_{k-s}) \\ 0 \leq u_v < u_v + l_v \leq k \ \forall v \\ l_1+\ldots+l_{k-s}=k-r}} Q_{u,l}^{r,s} \prod_{v=1}^{k-s} [z_{u_v}, z_{u_v+1}, \ldots, z_{u_v+l_v}]_g$$

$$(\forall r,s\colon 0 \leq r \leq s < k), \qquad (9)$$

$$A_s(z_0,\ldots,z_k)_g = \sum_{\substack{n,i \\ n=(n_1,\ldots,n_{k(k-1)/2}) \\ i=(i_1,\ldots,i_{k(k-1)/2}) \\ 0 \leq n_v < n_v + i_v \leq k \ \forall v \\ i_1+\ldots+i_{k(k-1)/2}=s+k(k-1)/2}} R_{n,i}^{s} \prod_{v=1}^{k(k-1)/2} [z_{n_v}, z_{n_v+1}, \ldots, z_{n_v+i_v}]_g \qquad (\forall s = 0,1,\ldots,k-1), \qquad (10)$$

where $Q_{u,l}^{r,s}$ and $R_{n,i}^{s}$ are homogeneous polynomials respectively of the orders $k - r$ and $s + k(k-1)/2$ in partial fractions of the form

$$(z_\mu - z_\lambda)/(z_h - z_t) \qquad (0 \leq h \leq \mu < \lambda \leq t \leq k),$$

and all coefficients of these polynomials are integers and are bounded by quantities depending only on k.

COROLLARY. If, in accordance with (9) and (10), the right-hand sides of the formulae (6) and (7) are considered as summatorial expressions, then the sum of orders of divided differences in the variable z contained as factors in any summand of the right-hand side of each of the formulae is equal to the order of the same kind (in respect to divided differences in the variable z) of the left-hand side of the formula under consideration: to the number k for (6) and to the number $k(k+1)/2$ for (7).

4. Moduli of smoothness. Let us fix a number $N \in [1,+\infty)$, a positive integer k, a point set $E \subset C$ and a function $f: E \to C$. For $z \in E$, $\delta > 0$ and integer $s = 0,1,\ldots,k$ one of concrete local moduli of smoothness is defined in [8] (see also [9], [3]) by the formula

$$\omega^{s}_{k,N,E,z}(f,\delta) = \sup_{z_0,\ldots,z_k} |[z_0,\ldots,z_k;f,z_0]|,$$

where the upper bound is taken over all simple point collections $z_0,\ldots,z_k \in E \cap \{\zeta : |\zeta - z| \le \delta\}$ satisfying the condition

$$\delta / |z_p - z_q| \le N \quad \forall p,q: s \le p < q. \tag{11}$$

If a locally rectifiable Jordan arc or curve Γ is taken as E, and $\rho_\Gamma(\zeta, z)$ is the curvilinear (in respect to Γ) distance between points $\zeta, z \in \Gamma$ then for a function $f: \Gamma \to C$ there is defined (see [3], [8], [9]) also the local modulus of smoothness

$$\widetilde{\omega}_{k,N,\Gamma,z}(f,\delta) = \sup_{z_0,\ldots,z_k} |[z_0,\ldots,z_k;f,z_0]|,$$

where the upper bound is taken over all simple point collections

$$z_0,\ldots,z_k \in \Gamma \cap \{\zeta: \rho_\Gamma(\zeta, z) \le \delta\} \tag{12}$$

situated in this order on some arc $\Gamma(z_0,\ldots,z_k) \subset \Gamma$ and satisfying the condition

$$\rho_\Gamma(z_i, z_{i+1}) / \rho_\Gamma(z_j, z_{j+1}) \le N \quad \forall i,j = 0,1,\ldots,k-1.$$

If Γ is a closed curve then for simplicity this definition may be restricted to point collections for which the ratio of length of $\Gamma(z_o,\dots,z_k)$ to the length of Γ is bounded by the quantity $k/(k+1)$.

Global moduli of smoothness $\omega^s_{k,N,E}(f,\delta)$ and $\widetilde{\omega}_{k,N,\Gamma}(f,\delta)$ are defined as upper bounds over $z\in E$ or $z\in\Gamma$ of corresponding local moduli of smoothness. Note that these moduli of the form ω^s_k (for simplicity other parameters are omitted) defined by means of the condition (11) are almost monotone (i.e. in order of magnitudes with respect to δ they are equivalent to increasing functions), and moduli of the form $\widetilde{\omega}_k$ are monotone in δ.

In (12) the quantity $\rho_\Gamma(\zeta, z)$ may be replaced by $|\zeta - z|$, and each of these quantities may be multiplied by positive constant (for example, $1/k$). In this respect we take into consideration that for lines Γ of the class S_λ, $1\leq\lambda< +\infty$ (for definition see [3], [2]), and in more general cases, the normality property is valid (see [3], [2]), and due to it under such changes orders of magnitudes of moduli of smoothness in respect to δ are preserved.

In [3], [8], [9] also other concrete types of moduli of smoothness were introduced and used. For instance, the condition (11) may be replaced by the condition

$$|(z_i - z_j)/(z_p - z_q)|\leq N \quad \forall i,\ j,\ p,\ q:\ s\leq p<q, \qquad (11')$$

(see [3], [8], [9]), and moduli obtained under this change are monotone in δ.

If $s = 0$ then the upper index in the notation ω^s_k may be omitted. Clearly, $\omega_k = \omega^o_k \leq \omega^1_k\leq \dots \leq \omega^k_k$. On any continuum (and on sets of essentially more general structure) ω^1_k is estimated from above by ω_k with constant coefficients not depending on δ.

<u>5. Inequalities with moduli of smoothness for function superpositions.</u> Let $G\subset C$ be a set containing more, than k, points, g be a function given on G and mapping it onto a set $F\subset C$, and constants a and b exist such that

$$0<1/b \leq |(g(z) - g(\zeta))/(z - \zeta)| \leq a < +\infty \quad \forall z, \zeta\in G.$$

Let a function f be given on F. Then there is true

THEOREM 4. For any $\delta > 0$ there hold the estimates

$$|\omega_{k,N,G,z}(f\circ g,\delta) - \omega_{k,Nab,F,g(z)}(f,a\delta)|$$

$$\leq c \sum_{j=1}^{k-1} \omega_{j,Nab,F,g(z)}(f,a\delta)\delta^{-j} \sum_{\substack{r_1,\dots,r_j \geq 1 \\ r_1+\dots+r_j=k}} \prod_{q=1}^{j} \omega_{r_q,N,G,z}(g,\delta), \quad (13)$$

$$|\omega_{k,N,F,w}(f,\delta) - \omega_{k,Nab,G,g^{-1}(w)}(f\circ g,b\delta)| \leq c\,\delta^{-k(k-1)/2}$$

$$\times \sum_{j=1}^{k-1} (\omega_{j,Nab,G,g^{-1}(w)}(f\circ g,b\delta)$$

$$\times \sum_{\substack{r_1,\dots,r_{k(k-1)/2} \geq 1 \\ r_1+\dots+r_{k(k-1)/2}=k(k+1)/2-j}} \prod_{q=1}^{k(k-1)/2} \omega_{r_q,Nab,G,g^{-1}(w)}(g,b\delta)) \quad (14)$$

where c depends only on k, N in (13) and only on k, N, a, b in (14).

Also global analogues of the estimates (13), (14) are true (see [1], [2]).

Suppose that a locally rectifiable Jordan arc or curve Γ is defined on a segment I of the real axis R by the natural equation $z = z(\sigma)$ (σ is the length of an arc on Γ, $\sigma = \sigma(z)$), and if Γ is a closed curve let us assume that I = R and the function $z(\sigma)$ is periodic with the period T_z corresponding to a single passing of Γ and equal to the length of Γ. Suppose that a finite function $\varphi(\sigma)$ is given on I, and if Γ is a closed curve then $\varphi(\sigma)$ is supposed to be periodic with the period T_z. Let $\omega_{k,I}(\varphi,\delta)$ be the usual (arithmetic and strongly centered - in terminology of [3]) modulus of smoothness of the function $\varphi(\sigma)$ of the real variable σ (not variable z). If Γ is a closed curve then for convenience sake this modulus will be considered only for values $\delta \leq (1/2)T_z$.

Note that $\omega_{k,I}(\varphi,\delta)$ turns out to be a quantity which is traditionally, by definition, assumed as a modulus $\omega_k(f,t)$ of smoothness of order k for the function $f(z) = (\varphi\circ\sigma)(z)$ on the line Γ (with the value $t = \delta/k$).

THEOREM 5. If Γ is a smooth line of the class C^{k-1} then

$$|\widetilde{\omega}_{k,1,\Gamma}(\varphi\circ\sigma,\delta) - \omega_{k,I}(\varphi,\delta)| \leq c \sum_{j=1}^{k-1} \omega_{j,I}(\varphi,\delta)\,\delta^{k-j}, \quad (15)$$

$$|\omega_{k,I}(\varphi,\delta) - \widetilde{\omega}_{k,1,\Gamma}(\varphi\circ\sigma,\delta)| \leq c \sum_{j=1}^{k-1} \widetilde{\omega}_{j,1,\Gamma}(\varphi\circ\sigma,\delta)\,\delta^{k-j} \quad (15')$$

where c depends only on k and Γ (more strictly, on a smoothness of Γ).

The result for a line Γ of the class S_λ, $1 \le \lambda < +\infty$, without the mentioned restriction on the smoothness of Γ is given in [1], [2].

The given theorems and other results of the work [1] have been essentially used by E.W. Karupu for the full solution of the problem of finite-difference smoothnesses of conformal homeomorphisms [10].

6. Relations between different moduli of smoothness and their application. Suppose $\lambda \geq 1$, $1 \le N \le M < +\infty$, k is a positive integer, Γ is a line of the class S_λ, f is a function continuous on Γ, $|\Gamma|$ is the length of Γ, and

$$E_{k-1}(f,\Gamma) = \inf_{P_{k-1}} \sup_{z\in\Gamma} |f(z) - P_{k-1}(z)|$$

is the best approximation of the function f on Γ by polynomials P_{k-1} of an order not exceeding k - 1. In [3], p. 227, there is proved the inequality

$$E_{k-1}(f,\Gamma) \le c\, \widetilde{\omega}_{k,2,\Gamma}(f,|\Gamma|) \tag{16}$$

with a constant c depending only on k and λ. In [1] there is given

THEOREM 6. There hold the estimates

$$\widetilde{\omega}_{k,M,\Gamma}(f,\delta) \le c\,\delta^{k-1} \int_{\delta\min\{N/M,1/2\}}^{\delta} \widetilde{\omega}_{k,N,\Gamma}(f,y)\, y^{-k} dy \quad (\delta > 0), \tag{17}$$

$$\widetilde{\omega}_{k,M,\Gamma}(f,\delta) \le c(M/N)^{k-1}\, \widetilde{\omega}_{k,N,\Gamma}(f,\delta) \quad (\delta > 0), \tag{18}$$

in which c depends only on k and λ.

From (16), (18) one can see that

$$E_{k-1}(f,\Gamma) \le c\, \widetilde{\omega}_{k,1,\Gamma}(f,|\Gamma|)$$

where c depends only on k and λ, and this is a complete analogue of Whitney's theorem [11]. The estimate (18) turned out to be new not only in the complex case, but also in the real one.

For $N \geqslant 2$ the estimates (17), (18) in the former shape were obtained in [3], p. 219 - 224, then in a more general form (and for most of sets in C) - in [8],[9] where a simpler method not connected with normality property for moduli of smoothness was used. In order to replace the constant N = 2 by the constant N = 1, we made use of additional considerations generalizing one Whitney's idea (and his result) from [11] where it was applied to arithmetical moduli of smoothness for functions of a real variable. Besides for this purpose we established and used the following statements.

LEMMA. Whatever be simple point collections $z_o,\dots,z_k$ and $\zeta_1, \dots, \zeta_k$ and a finite function f defined at these points, there hold

$$[z_o,\dots,z_k]_f = \sum_{j=0}^{k} [z_j, \zeta_1,\dots,\zeta_k;f,z_j] \prod_{\substack{i=0\\ i\neq j}}^{k} (z_j - z_i)^{-1}.$$

Note that if z_j coincides with one of the points $\zeta_1,\dots,\zeta_k$ then it is supposed that $[z_j, \zeta_1,\dots,\zeta_k;f,z_j] = 0$.

COROLLARY. There exist $j \in [0,k]$ such that

$$|[z_j, \zeta_1,\dots,\zeta_k;f,z_j]|$$

$$\geqslant \left| \frac{[z_o,\dots,z_k;f,z_s]}{k+1} \Big(\prod_{\substack{i=0\\ i\neq j}}^{k} (z_j-z_i)\Big)\Big(\prod_{\substack{t=0\\ t\neq s}}^{k} (z_s-z_t)^{-1}\Big) \right| \quad \forall s = 0,\dots,k.$$

On the basis of the theorem 6 and normality property for moduli of smoothness on lines of the class S_λ, the lemma 5.7.1 of [3] (see also inequality (9) of [8]) obviously contains the following

COROLLARY. Under the conditions of the theorem 6 for any finite N there holds the estimate

$$\omega^{1}_{k,N,\Gamma}(f,\delta) \leqslant c\, \widetilde{\omega}_{k,1,\Gamma}(f,\delta) \quad (\delta > 0) \qquad (19)$$

where the constant c depends only on k, N and λ.

This is an extension and strengthening of above mentioned Whitney's theorem (even in the real case).

The following statement is proved as well (for $\Gamma \in S_\lambda$).

THEOREM 7. If $k\lambda \leqslant N < +\infty$ then in respect to δ there holds the order equality

$$\omega_{k,N,\Gamma}(f,\delta)\asymp\widetilde{\omega}_{k,1,\Gamma}(f,\delta). \tag{20}$$

From the estimates (16) - (20) it is seen that a number of results of [3] and other works, containing estimates for different entities via uniform moduli of smoothness, remain valid under replacement of these moduli by $\widetilde{\omega}_{k,1,\Gamma}$. For instance, this regards the estimates (5.7.2), (5.7.2′), (5.7.15) of [3] and many other results.

Analogues of the results of this work hold also for centered, strongly centered and other types of moduli (see [3], [8], [9]).

References

1. Тамразов П.М. Конечно-разностные тождества и оценки модулей гладкости суперпозиций функций. Препринт 77-5. Институт математики АН УССР, Киев, 1977.
2. Tamrazov P.M. Structural and approximational properties of functions in the complex domain. In: Linear Spaces and Approximation, Birkhäuser Verlag, Basel, 1978, 503 - 514.
3. Тамразов П.М. Гладкости и полиномиальные приближения. Наукова думка, Киев, 1975.
4. Warschawski S.E. On the higher derivatives at the boundary in conformal mappings. Trans. Amer. Math. Soc., 38 (1935), 310 - 340.
5. Ковальчук Р.Н. Об одном обобщении теоремы Келлога. Укр. матем. ж., 17 (1965), 104 - 108.
6. Колесник Л.И. Обращение теоремы типа Келлога. Укр. матем. ж., 21 (1969), 104 - 108.
7. Карупу Е.В. О конечно-разностных гладкостях конформных отображений. Препринт ИМ-77-6. Институт математики АН УССР, Киев, 1977.
8. Тамразов П.М. Конечно-разностные гладкости и полиномиальные приближения. Препринт 75-10. Институт математики АН УССР, Киев, 1975.
9. Tamrazov P.M. Finite-difference smoothnesses and approximation. Colloquia Mathematica Societatis János Bolyai. 19. Fourier Analysis and Approximation Theory, Budapest, 1976, 827 - 843.
10. Карупу Е.В. Конечно-разностные гладкости конформных гомеоморфизмов. - В кн.: Модули гладкости в задачах теории функций. Препринт 78-4. Институт математики АН УССР, Киев, 1978.
11. Whitney H. On functions with bounded n^{th} differences. Journ. Mathem. Pures et Applic., 36 (1957), N 1, 67 - 95.

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