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DEGREE OF APPROXIMATION OF HERMITE-FEJER INTERPOLATION BASED ON THE ZEROS OF LEGENDRE POLYNOMIAL

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1. Introduction Let $f(x)\in C_{[-1,1]}$ and let $-1<x_{n,n}<x_{n-1,n}<\cdots<x_{1,n}<1$ be the zeros of Legendre polynomial $P_n(x)$ of degree n. It is well known that Hermite-Fejér interpolation polynomial based on $\{x_{k,n}\}_{k=1}^{n}$ is

$$H_n(f,x)=\sum_{k=1}^{n} f(x_{k,n})\,\frac{1-2xx_{k,n}+x_{k,n}^2}{1-x_{k,n}^2}\,l_{k,n}^2(x), \tag{1.1}$$

where

$$l_{k,n}(x)=\frac{P_n(x)}{(x-x_{k,n})P'_n(x_{k,n})}.$$

On the degree of approximation of Hermite-Fejér interpolation, J. Szabados[1] has improved the results of L. Fejér[2] and A. Schonhage[3]. He proved

$$|f(x)-H_n(f,x)|=\max\left\{\left|f(\pm1)-\frac{1}{2}\int_{-1}^{1}f(t)dt\right|\right\}O\left(\frac{1}{n\sqrt{1-x^2}}\right)+O\left(\omega\left(f,\frac{\ln n}{n}\right)\right)$$

for $|x|<1$, where $\omega(f,\delta)$ is the modulus of continuity of f. Recently, N. Misra[4] improved above statement by giving better estimate for the degree of approximation. First of all, Misra pointed out

$$|f(x)-H_n(f,x)|=\max\left\{\left|f(\pm1)-\frac{1}{2}\int_{-1}^{1}f(t)dt\right|\right\}P_n^2(x)$$

$$+O\left\{P_n^2(x)\omega\left(f,\frac{1}{n}\right)+\sum_{i=1}^{n}\frac{1}{i^2}\,\omega\left(f,\frac{i(1-x^2)^{3/4}|P_n(x)|}{\sqrt{n}}\right)\right\} \tag{1.2}$$

for $|x|\leqslant1$. Further, he showed that there exists a function $f\in\mathrm{Lip}\,1$ and a constant C such that

$$|f(0)-H_n(f,0)|\geqslant C\frac{\ln n}{n}$$

whenever n is even integer. Hence at the point $x=0$ the estimation (1.2) is precise for the class Lip 1 and even n.

Obviously, it is natural to ask what esults would be attached to the cases $x\neq0$ or odd n and for the class $H_\omega=\{f:\omega(f,\delta)\leqslant\omega(\delta)\}$, where $\omega(\delta)$ is a given modulus of continuity. Of the main aim of this paper, we shall give an integral answer of above problems. First of all, we shall give a precise estimation for the quantity

$$\sup_{f\in H_\omega}|f(x)-H_n(f,x)|,$$

and further, for each $x\in[-1,1]$, we shall prove that there exists a function $f_x\in\mathrm{Lip}_1 1$ satisfying the relation

$$|f_x(x)-H_n(f_x,x)|\geqslant C\{P_n^2(x)+(1-x^2)P_n^2(x)\ln n\},$$

where and in sequel by C we denote the absolute constants which are, in general, different in different folmulas. Finally, the improvement of (1,2) has been obtained. In the last section of this paper, we shall discuss the degree of approximation by Egervary-Turan polynomial and improve the result of J. Prasad and R. B. Saxena[5].

2. Preliminaries Let $\{a_{k,n}(x)\}$ and $\{b_{k,n}(x)\}$ be the functions defined on $[-1,1]$. The notation $a_{k,n}(x)\sim b_{k,n}(x)$ means that there exists a positive r such that

$$ra_{k,n}(x)\leqslant b_{k,n}(x)\leqslant\frac{1}{r}a_{k,n}(x)$$

holds for $x\in[-1,1]$ and integers k and n. Using this notation we write some properties of $P_n(x)$, which will be of use, as follows (See[6])

$$|P_n'(x_{k,n})|\sim k^{-3/2}n^2\quad\left(k=1,2,\cdots,\left[\frac{n}{2}\right]\right),\tag{2.1}$$

$$|P_n'(x_{k,n})|\sim(n+1-k)^{-3/2}\quad\left(k=\left[\frac{n}{2}\right]+1,\cdots,n\right).\tag{2.2}$$

If $x_{k,n}=\cos\theta_{k,n}$, then

$$\left(k-\frac{1}{2}\right)\pi<\theta_{k,n}\left(n+\frac{1}{2}\right)<k\pi\quad(k=1,2,\cdots,n).\tag{2.3}$$

For $x\in[-1,1]$, we have

$$(1-x^2)^{1/4}|P_n(x)|\leqslant\left(\frac{2}{n\pi}\right)^{1/2},\tag{2.4}$$

$$|P_n(x)|\leqslant1,\tag{2.5}$$

$$(1-x^2)^{3/4}|P_n'(x)|\leqslant C\sqrt{n}.\tag{2.6}$$

Let $-1<x_{n-1,n}^*<x_{n-2,n}^*<\cdots<x_{1,n}^*<1$ be the zeros of $P_n'(x)$, then

$$|P_n(x_{k,n}^*)|\sim\left(n\sin\frac{k}{n}\pi\right)^{-1/2}\quad(k=1,2,\cdots,n-1).\tag{2.7}$$

For each $x\in[-1,1]$, we denote by $x_{j,n}$ the zero of $P_n(x)$, which is nearest to x, and write

$$I_j=\frac{\omega(|x-x_{j,n}|)(1-x^2)P_n^2(x)}{|x-x_{j,n}|^2(1-x_{j,n}^2)P_n'^2(x_{j,n})}.$$

Lemma If $\omega(\delta)$ is a given modulus of continuity, then inequality

$$I_j\leqslant C\omega\left(\frac{(1-x^2)^{3/4}|P_n(x)|}{\sqrt{n}}\right)\tag{2.8}$$

holds uniformly on $[-1,1]$.

Proof Without loss of generality, we may assume that $x\in[0,1]$. If $x\in\left[0,\cos\frac{\theta_1}{2}\right]$, then there exists a point ξ which lies between $x_{j,n}$ and x such that $P_n(x)=(x-x_{j,n})P_n'(\xi)$, and by (2.3), we have

$$1-x^2\sim1-\xi^2\sim1-x_{j,n}^2\sim\frac{j^2}{n^2}.\tag{2.9}$$

Hence it follows from (2.1) and (2.6) that

$$\frac{(1-x^2)P_n^2(x)}{|x-x_{j,n}|^2(1-x_{j,n}^2)P_n'^2(x_{j,n})}\leqslant\frac{(1-x^2)P_n'^2(\xi)}{(1-x_{j,n}^2)P_n'^2(x_{j,n})}<C,$$

where $C>1$. Therefore the well known inequality

$$a\omega(\delta)\leqslant2\omega(a\delta)\quad(0<a\leqslant1)\tag{2.10}$$

and (2.4) imply

$$I_j \leqslant C\omega\left(\frac{(1-x^2)^{3/4}|P_n(x)|}{\sqrt{n}}\right). \tag{2.11}$$

If $x \in \left[\cos\frac{\theta_1}{2}, 1\right]$, then

$$1-x^2 \leqslant \frac{C}{n^2} \text{ and } x-x_1 \sim \frac{1}{n^2}.$$

Thus from (2.1), (2.5) and (2.10) we get

$$I_j \leqslant C\omega\left(\frac{1}{n^2}\right)n^2(1-x^2)P_n^2(x) \leqslant C\omega\left(\frac{(1-x^2)^{3/4}|P_n(x)|}{\sqrt{n}}\right), \tag{2.12}$$

Combining (2.11) and (2.12) we obtain (2.8). Lemma is proved.

3. Theorems Let $x \in [-1,1]$, we write

$$A_n(\omega,x) = I_j + \sum_{k=1}^{n} \omega(\Delta_{n,k}(x))\frac{(1-x^2)P_n^2(x)}{k\Delta_{n,k}(x)},$$

for the sake of convenience, where $\Delta_{n,k}(x) = \frac{k}{n}\left(\sqrt{1-x^2}+\frac{k}{n}\right)$.

Theorem 1 If $\omega(\delta)$ is a given modulus of continuity, then

$$\sup_{f\in H_\omega} |f(x)-H_n(f,x)| \sim \omega(1)P_n^2(x) + A_n(\omega,x) \tag{3.1}$$

holds uniformly on $[-1,1]$

Proof We have from (1.1)

$$H_n(f,x) = \sum_{k=1}^{n} f(x_{k,n})\frac{P_n^2(x)}{(1-x_{k,n}^2)P_n'^2(x_{k,n})} + \sum_{k=1}^{n} f(x_{k,n})\frac{1-x^2}{1-x_{k,n}^2}l_{k,n}^2(x).$$

Hence

$$f(x)-H_n(f,x) = \sum_{k=1}^{n}(f(x)-f(x_{k,n}))\frac{P_n^2(x)}{(1-x_{k,n}^2)P_n'^2(x)} + \sum_{k=1}^{n}(f(x)-f(x_{k,n}))\frac{1-x^2}{1-x_{k,n}^2}l_{k,n}^2(x). \tag{3.2}$$

Therefore it is easy to see that

$$\sup_{f\in H_\omega}|f(x)-H_n(f,x)| = \sum_{k=1}^{n}\omega(|x-x_{k,n}|)\frac{P_n^2(x)}{(1-x_{k,n}^2)P_n'^2(x_{k,n})} + \sum_{k=1}^{n}(f(x)-f_{k,n})\frac{1-x^2}{1-x_{k,n}^2}l_{k,n}^2(x)$$
$$= \Sigma_1 + \Sigma_2. \tag{3.3}$$

First we shall estimat Σ_1. Setting

$$\Sigma_1 = \left(\sum_{k=1}^{[\frac{n}{2}]} + \sum_{k=[\frac{n}{2}]+1}^{n}\right)\omega(|x-x_{k,n}|)\frac{P_n^2(x)}{(1-x_{k,n}^2)P_n'^2(x_{k,n})} = J_1 + J_2. \tag{3.4}$$

Obviously, it follows from (2.1) and (2.4) that

$$J_1 \sim \sum_{k=1}^{[\frac{n}{2}]}\omega(|x-x_{k,n}|)\frac{k}{n^2}P_n^2(x) \leqslant \omega(1)P_n^2(x). \tag{3.5}$$

However, for $x \in \left[\frac{\sqrt{2}}{2}, 1\right]$ and $k = \left[\frac{3}{8}n\right]+1, \cdots \left[\frac{n}{2}\right]$, we have $|x-x_{k,n}| \geqslant \frac{1}{10}$,

so that

$$J_1 \geqslant \frac{\omega(1)}{100}P_n^2(x). \tag{3.6}$$

Similary, for $x \in \left[0, \frac{\sqrt{2}}{2}\right]$ we have also (3.6). Combining (3.5) and (3.6) we obtain

$$J_1 \sim \omega(1)P_n^2(x). \tag{3.7}$$

Similary, we have $J_2 \sim \omega(1)P_n^2(x)$ and it follows from (3.7) and (3.4) that

$$\Sigma_1 \sim \omega(1) P_n^2(x). \tag{3.8}$$

In order to estimate Σ_2 we write

$$\Sigma_2 = I_j + \left(\sum_{k=1}^{j-1} + \sum_{k=j+1}^{[\frac{7}{8}n]} + \sum_{k=[\frac{7}{8}n]+1}^{n} \right) \frac{\omega(|x-x_{k,n}|)(1-x^2) l_k^2(x)}{1-x_{k,n}^2} = I_j + J_3 + J_4 + J_5, \tag{3.9}$$

where $J_3 = 0$ if $j=1$. Without lcos the generality we may assume that $x \in [0,1]$, so that $j \leqslant \left[\frac{n}{2}\right]$. If $j>1$ and $k \leqslant j-1$, then

$$|x-x_{k,n}| = 2\sin\frac{\theta-\theta_k}{2}\sin\frac{\theta+\theta_k}{2} \sim \frac{j(j-k)}{n^2}.$$

Hence from (2.1) and (2.9) we get

$$J_3 \sim \sum_{k=1}^{j-1} \omega\left(\frac{j(j-k)}{n^2}\right) \frac{(1-x^2) P_n^2(x) n^2 k}{j^2 (j-k)^2} \leqslant \sum_{k=1}^{j-1} \omega\left(\frac{jk}{n^2}\right) \frac{(1-x^2) P_n^2(x) n^2}{jk^2}.$$

But in this case $\frac{k^2}{n^2} \leqslant \frac{kj}{n^2} < C\frac{k}{n}\sqrt{1-x^2}$, so that

$$J_3 \leqslant C \sum_{k=1}^{j-1} \omega(\Delta_{k,n}(x)) \frac{(1-x^2) P_n^2(x)}{k \Delta_{k,n}(x)}. \tag{3.10}$$

If $k \geqslant j+1$, then

$$|x-x_{k,n}| = \sin\theta \sin(\theta_k-\theta) + 2\cos\theta \sin^2\frac{\theta_k-\theta}{2} \sim \sqrt{1-x^2}\frac{k-j}{n} + \left(\frac{k-j}{n}\right)^2.$$

Hence from (2.1) and (2.3) we get

$$J_4 \sim \sum_{k=j+1}^{[\frac{7}{8}n]} \omega(\Delta_{k-j,n}(x)) \frac{(1-x) P_n^2(x)}{(k-j)\Delta_{k-j,n}(x)} = \sum_{k=1}^{[\frac{7}{8}n]-j} \omega(\Delta_{k,n}(x)) \frac{(1-x^2) P_n^2(x)}{k\Delta_{k,n}(x)}.$$

But, it is easily verified that

$$\sum_{k=[\frac{7}{8}n]+j}^{n} \omega(\Delta_{k,n}(x)) \frac{(1-x^2) P_n^2(x)}{k\Delta_{k,n}(x)} \leqslant C \sum_{k=1}^{[\frac{7}{8}n]-j} \omega(\Delta_{k,n}(x)) \frac{(1-x^2) P_n^2(x)}{k\Delta_{k,n}(x)}.$$

Therefore we have

$$J_4 \sim \sum_{k=1}^{n} \omega(\Delta_{k,n}(x)) \frac{(1-x^2) P_n^2(x)}{k\Delta_{k,n}(x)}. \tag{3.11}$$

Similaly, we have $J_5 \leqslant CJ_4$, thus substituting (3.10) and (3.11) in to (3.9) we find

$$\Sigma_2 \sim I_j + \sum_{k=1}^{n} \omega(\Delta_{k,n}(x)) \frac{(1-x^2) P_n^2(x)}{k\Delta_{k,n}(x)}. \tag{3.12}$$

Combining (3.8), (3.12) and (3.3) it follows that (3.1) holds. This completes the proof of Theorem 1.

Obviously, if $f \in \text{Lip}_1 1$, then (2.1) and (2.9) imply

$$I_j \sim \frac{\sqrt{1-x_{jn}^2}(1-x^2) P_n^2(x)}{n|x-x_{j,n}|}.$$

Thus Theorem 1 implies following

Theorem 2 For class $\text{Lip}_1 1$ we have

$$\sup_{f \in \text{Lip}_1 1} |f(x) - H_n(f,x)| \sim P_n^2(x)(1+(1-x^2)\ln n) + \frac{\sqrt{1-x_{jn}^2}(1-x^2) P_n^2(x)}{n|x-x_{j,n}|}$$

holds uniformly on $[-1,1]$.

Further, using (2.7) and Theorem 2 we obtain

$$\sup_{f \in \text{Lip}_1 1} |f(x_{\nu,n}^*) - H_n(f, x_{\nu,n}^*)| \sim \frac{1}{n\sin\frac{\nu}{n}\pi} + \frac{\ln n}{n}\sin\frac{\nu}{n}\pi$$

where $\nu=1,2,\cdots,n-1$.

Theorem 3 If $f\in H_\omega$, then

$$f(x)-H_n(f,x)=\left\{f(x)-\frac{1}{2}\int_{-1}^{1}f(t)dt\right\}P_n^2(x)+O\left(\omega\left(\frac{1}{n}\right)P_n^2(x)+A_n(\omega,x)\right)$$

holds unformly on $[-1,1]$.

Proof We have from (1.1)

$$f(x)-H_n(f,x)=\left(f(x)-\frac{1}{2}\int_{-1}^{1}f(t)dt\right)P_n^2(x)+\left(\frac{1}{2}\int_{-1}^{1}f(t)dt-\sum_{k=1}^{n}\frac{f(x_{k,n})}{(1-x_{k,n}^2)P_n'^2(x_{k,n})}\right)P_n^2(x)$$

$$+\frac{f(x)-f(x_{k,n})}{1-x_{k,n}^2}(1-x^2)l_{k,n}^2(x),$$

But in the proof of Theorem 1 we had proved

$$\sum_{k=1}^{n}\frac{f(x)-f(x_{k,n})}{1-x_{k,n}^2}(1-x^2)l_{k,n}^2(x)=O(A_n(\omega,x)).$$

Hence by the well known estimation (See[4])

$$\left|\frac{1}{2}\int_{-1}^{1}f(t)dt-\sum_{k=1}^{n}\frac{f(x_{k,n})}{(1-x_{k,n}^2)P_n'^2(x_{k,n})}\right|\leqslant C\omega\left(f,\frac{1}{n}\right)P_n^2(x),$$

we obtain

$$f(x)-H_n(f,x)=\left(f(x)-\frac{1}{2}\int_{-1}^{1}f(t)dt\right)P_n^2(x)+O\left(\omega\left(\frac{1}{n}\right)P_n^2(x)+A_n(\omega,x)\right),$$

and Theorem 3 is proved.

It is no difficult to verify that

$$|f(x)-f(\operatorname{sign}x)|P_n^2(x)\leqslant 2\omega(f,(1-x^2)P_n^2(x))\leqslant C\omega\left(\frac{(1-x^2)^{3/4}|P_n(x)|}{\sqrt{n}}\right)$$

and

$$\omega(f,\Delta_{k,n}(x))\frac{(1-x^2)P_n^2(x)k}{\Delta_{n,k}(x)}\leqslant C\omega\left(f,\frac{k(1-x^2)^{3/4}|P_n(x)|}{\sqrt{n}}\right).$$

Applying Lemma and Theorem 3, we obtain

Corollary If $f\in C_{[-1,1]}$, then

$$f(x)-H_n(f,x)=\left(f(\operatorname{sign}x)-\frac{1}{2}\int_{-1}^{1}f(t)dt\right)P_n^2(x)+O\left\{\omega\left(f,\frac{1}{n}\right)P_n^2(x)+\sum_{k=1}^{n}\frac{1}{k^2}\omega\left(f,\frac{k(1-x^2)^{3/4}|P_n(x)|}{\sqrt{n}}\right)\right\}$$

holds uniformly on $[-1,1]$.

4. Applications Now we can apply the simulation discussing in Section 2 to the Egervary-Turan polynomial (See[5])

$$Q_n(f,x)=\left(\frac{1+x}{2}f(1)+\frac{1-x}{2}f(-1)\right)P_n^2(x)+\sum_{k=1}^{n}f(x_{k,n})\frac{1-x^2}{1-x_{k,n}^2}l_{k,n}^2(x).$$

Obviously,

$$\omega(1-x)\frac{1+x}{2}+\omega(1+x)\frac{1-x}{2}\sim\omega(1-x^2).$$

Hence from (3.12) we obtain the following

Theorem 4 Let $\omega(\delta)$ is a given moddulus of continuity, then

$$\sup_{f\in H_\omega}|f(x)-Q_n(f,x)|\sim\omega(1-x^2)P_n^2(x)+A_n(\omega,x),$$

$$\sup_{f\in \mathrm{Lip}_1 1}|f(x)-Q_n(f,x)|\sim(1-x^2)P_n^2(x)\ln n+\frac{\sqrt{1-x_{j,n}^2}(1-x^2)P_n^2(x)}{n|x-x_{j,n}|}$$

hold uniformly on $[-1,1]$.

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