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GENERALIZED-ANALYTIC SETS IN THE PARTS OF MAXIMAL IDEAL SPACE

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In [1] A. Browder proved the following generalization of a theorem of A. Gleason [2] : If A is a commutative Banach algebra with unit and Θ is a linear multiplicative functional on A with finite dimensional space of continuous point derivations, then some metric neighbourhood of Θ in the maximal ideal space sp A of A is homeomorphic to an analytic subset of certain finite dimensional polydisc and the Gelfand transforms of algebra elements are analytic functions on it. In [4] Gleason's theorem was generalized to the case of countable generated maximal ideals and in [6] - to the case of maximal ideals, generated by countable semigroups. In both cases the usual analyticity was replaced by analyticities of generalised nature, namely - of infinite many variables - in the first case, and by Arens-Singer's analyticity - in the second. Here we give some sufficient conditions that guarantée the "living" of analytic structure in Arens-Singer sense within a part of the maximal ideal space.

Let Γ be an additive subgroup of real rational numbers provided with discrete topology, $G = \hat{\Gamma}$ be the compact connected group of characters of the group Γ and Δ_G be the "big disc", i.e. the cone $[0,1).G = G \times [0,1)/G \times \{0\}$ with $* = G \times \{0\}/G \times \{0\}$ as a peak. For any $p \in \Gamma_+ = \Gamma \cap [0,\infty)$ the corresponding character $\chi_p(g) = g(p)$ of G is extendable to the whole big disc $\overline{\Delta}_G$ in the following way: $\chi_p(\lambda,g) = \lambda^p \chi_p(g)$ for $p \neq 0$, $\lambda \neq 0$; $\chi_p(*) = 0$ for any $p \neq 0$ and $\chi_0 \equiv 1$. In 1956 R. Arens and I. Singer [5] introduced the algebra A_G of generalized analytic functions on a compact group with dual group, which is ordered. The following is an equivalent definition: a continuous function f on $\overline{\Delta}_G$ is <u>generalized-analytic</u> iff it is approximated uniformly on $\overline{\Delta}_G$ by linear combinations over $\mathbb{C}$ of functions χ_p (generalized polynomials). If $\Delta_G(\varepsilon)$ denotes the big disc with radius ε, i.e. the set $[0,\varepsilon).G = G\times[0,\varepsilon)/G\times\{0\}$, then by $A_G(\varepsilon)$ we shall denote the uniform algebra of continuous on $\overline{\Delta}_G(\varepsilon)$ functions that are uniformly approximable on $\overline{\Delta}_G(\varepsilon)$ by generalized polynomials.

Let A be a uniform commutative Banach algebra with unit. The maximal ideal space of A, provided with Gelfand (weak*-) topology, we denote by $\operatorname{sp} A$. Remember that the parts of $\operatorname{sp} A$ are the equivalence classes of relation $\|\varphi - \theta\| < 2$. Let $z = (z_1, z_2, \dots)$ and I be the set of sequences $\alpha = (\alpha_1, \alpha_2, \dots)$, where α_j are nonnegative integers, finite of which are different from zero. As usual $|\alpha| = \sum |\alpha_j|$ and $z^\alpha = z_1^{\alpha_1}.z_2^{\alpha_2}. \dots$ for any $z \in \mathbb{C}^\infty$ and $\alpha \in I$. Analogously, if $a = \{a_j\}_1^\infty$ is a sequence of elements of A and $\alpha \in I$, by a^α we denote the finite product $a_1^{\alpha_1}.a_2^{\alpha_2}. \dots$. A polynomial of infinite dimensional argument is any linear combination of functions z^α, $\alpha \in I$.

For a given linear subspace $E \subset A$ we denote by B_E the closed unit ball of E. Let $\theta \in \operatorname{sp} A$ and M be the kernel of θ. Suppose that there exists a sequence $u = \{u_j\}_1^\infty$, $\|u_j\| = 1$ with elements of M, such that every element $f \in M$ can be presented in a unique way in the form

$$f = \sum_{j=1}^{\infty} \lambda_j u_j + g, \tag{1}$$

where $\lambda_j \in \mathbb{C}$, $j = 1, 2, \dots$, $\sum |\lambda_j| < \infty$ and $g \in [M^2]$. Let $\{C_\alpha\}$, $\alpha \in I$, be a set of positive numbers, satisfying the following reccurent dependence:

$$C_\alpha = \frac{1}{2\varrho} \geq 1 \quad \text{if } |\alpha| = 1\ ; \quad C_\alpha = \frac{1}{2\varrho} \sum_{\beta+\gamma=\alpha} C_\beta C_\gamma \quad \text{for } |\alpha| = 2.$$

<u>Lemma 1</u>. Let N denotes the linear subspace of M:

$$N = \Big\{ f \in M \;\Big|\; f = \sum_{k=1}^{\infty} \mu_k g_k h_k \ ;\ g_k, h_k \in B_M,\ \|\{\mu_k\}\|_{l_1} < \infty \Big\}.$$

There exists a constant $\varrho : 0 < \varrho \leq \frac{1}{2}$, such that if r is a positive integer, the elements f of B_M take the form:

$$f = \sum_{|\alpha| \leq r} \xi_\alpha (\lambda u)^\alpha + F$$

for some constants ξ_α, $|\xi_\alpha| = C_\alpha$ for $|\alpha| \leq r$ and λ_j, $j = 1, 2, \dots$ with $\sum |\lambda_j| \leq \frac{1}{2\varrho}$, and for some element $F \in N$ with $|\varphi(F)| \leq \varrho^{r+1}$

for any $\varphi \in sp\,A$, such that $\|\varphi - \Theta\| \leq \varepsilon = \varrho^5$.

The proof is based essentially on the open mapping theorem. As an immediate corollary from the case $r = 1$ of this Lemma we obtain that $N = [M^2]$, i.e. that the space $M/[M^2] = M/N$ of continuous point derivations in Θ is not finite dimensional.

Let $\Gamma = \mathbb{Q}^\nu = \mathbb{Q} \cap [\nu, +\infty) \cup \{0\}$, where $\nu > 0$. For a given subset E of the big disc $\overline{\Delta}_{\widehat{\mathbb{Q}}^\nu}(\eta)$ by $A^\nu(E)$ we denote the algebra of uniform limits on E of linear combinations of functions χ_p with p belonging only to $\mathbb{Q}^\nu$. In what follows by $u: sp\,A \to \mathbb{C}^\infty$ will be denoted the function $\hat{u}(\varphi) = (\hat{u}_1(\varphi), \hat{u}_2(\varphi), \ldots) = (\varphi(u_1), \varphi(u_2), \ldots)$, $u_j \in A$. If V is a subset of some $\Delta_G(\eta)$, it will be called <u>generalized-analytic set</u> if it coincides with the vanishing set of some family of generalized-analytic functions on $\overline{\Delta}_G(\eta)$.

<u>T h e o r e m 2.</u> Let A be a uniform algebra and Θ be a fixed linear multiplicative functional of A. Suppose that there exists a multiplicative semigroup $\{u_{p(j)}\}_{j=1}^\infty$ in $M = \operatorname{Ker}\Theta$, $\|u_{p(j)}\| \leq 1$, isomorphic to the additive semigroup $\mathbb{Q}^\nu$ for some ν : $0 < \nu < 1$, such that the algebra A_0 of elements $f \in A$, presentable uniquely in the form $f = \Theta(f) + \sum_{p(j) \in [\nu, 2\nu)}^{\infty} \lambda_j\, u_{p(j)} + g$, $\sum_1^\infty |\lambda_j| < \infty$,

$g \in N = \{f \in M\; f = \sum_1^\infty \mu_k\, g_k\, h_k,\ g_k, h_k \in B_M,\ \|\{\mu_k\}\|_{l_1} < \infty\}$ is dense in A. Then there exist a set $U \subset sp\,A$, containing Θ as an inner point in a metric topology in A_0^*, a generalized-analytic set V in some big disc $\Delta_G(d)$, $G = \widehat{\mathbb{Q}}$, $d > 0$, and a homeomorphism $\tau : U \to V$, $\tau(\Theta) = *$, such that $\hat{f} \circ \tau^{-1}$ is a generalized-analytic function belonging to $A^\nu(V)$ for any $f \in A$.

It is clear that under above conditions the space $M/[M^2] = [A_0 \cap M]/[M^2]$ of continuous point derivations has countable many dimensions together with the space M/N.

P r o o f: Provided with the norm $\|f\|_0 = |\Theta(f)| + \sum_1^\infty |\lambda_j| + \|g\|$, A_0 is a commutative Banach algebra with maximal ideal space $sp\,A$. Moreover $A_0 \cong \mathbb{C} \oplus l_1(\{u_{p(j)} + [M^2]\}_1^\infty) + [M^2]$. The standard dual norm of $(A_0, \|\ \|_0)^*$ we denote by $\|\ \|^0$. If $\psi \in sp\,A$, then for every $p = n/m \in \mathbb{Q}^\nu$ it holds:

$$|\psi(u_p)|^m = |\psi(u_{n/m})|^m = |\psi(u_n)| = |\psi(u_1)|^n, \text{ i.e.}$$
$$|\psi(u_p)| = |\psi(u_1)|^{n/m} = |\psi(u_1)|^p.$$

Let ϱ de the positive number from Lemma 1 and $\varepsilon = \varrho^5$, $\eta = \left(\frac{2\varrho^7}{1+\varrho^4}\right)^{1/\nu} < \varepsilon$.

We define the set

$$U_1 = \{\varphi \in \mathrm{sp}\, A \,|\, \|\varphi - \Theta\|^o \leq \varepsilon,\ |\varphi(u_1)| \leq \eta\},$$

considered with the weak*- topology induced on U_1 from sp A and claim that for every $\varphi \in U_1$ there exists a point $\tau(\varphi)$ in $\overline{\Delta}_G(\eta)$, $G = \hat{\mathbb{Q}}$, such that $\chi_p(\tau(\varphi)) = \varphi(u_p) = \hat{u}_p(\varphi)$ for any $p \in \mathbb{Q} \cap [\nu, 2\nu)$. Let $\lambda_\varphi = |\varphi(u_1)| \leq 1$ and $\gamma_\varphi(p) = \varphi(u_p)/|\varphi(u_p)|$ for $p \in \mathbb{Q} \cap [\nu, 2\nu)$, $\gamma_\varphi(p) = \overline{\gamma_\varphi(-p)}$ for $p \in -\mathbb{Q} \cap [\nu, 2\nu)$. If $\varphi \in U_1$, the point $\tau(\varphi) = (\lambda_\varphi, \gamma_\varphi) \in \overline{\Delta}_G(\eta)$ and

$$\chi_p(\tau(\varphi)) = \lambda_\varphi^p \chi_p(\gamma_\varphi) = \lambda_\varphi^p \gamma_\varphi(p) = |\varphi(u_p)|(\varphi(u_p)/|\varphi(u_p)|) = \\ = \varphi(u_p) = \hat{u}_p(\varphi).$$

Consequently $\tau(U_1) \subset \overline{\Delta}_G(\eta)$ and the point $(\lambda_\varphi, \gamma_\varphi)$ satisfies the claimed statement. The point $\tau(\varphi)$ is uniquely defined for the functions χ_p, $p \in \mathbb{Q} \cap [\nu, 2\nu)$, separate the points of $\overline{\Delta}_G(\eta)$. One can see directly that τ is a continuous mapping. From $\chi_p(\tau(\varphi)) = \hat{u}_p(\Theta) = 0$ for any $p \in \mathbb{Q} \cap [\nu, 2\nu)$ we see that $\tau(\Theta) = *$. Applying Lemma 1 to the algebra $(A_o, \|\ \|_o)$, we obtain that given a function $f \in M_o = A_o \cap M$, there exists a sequence $\{p_r | p_r(\lambda z) = \sum_{|\alpha| \leq r} \xi_{\alpha,r} \lambda^\alpha z^\alpha, |\xi_{\alpha,r}| \leq 2\|f\|_o C_\alpha\}$ of polynomials of countable many variables, for which the functions $p_r(\lambda \hat{u})$ approximate uniformly on U_1 the function $f - \Theta(f)$. Consequently every $\varphi \in U_1$ is uniquely determined by its values on the elements χ_p, $p \in \mathbb{Q}^\nu$. An immediate consequence of this result is that the mapping $\tau: U_1 \to \tau(U_1) \subset \overline{\Delta}_G(\eta)$ is one-to-one. As an one-to-one and continuous mapping from a locally compact set to a Hausdorff space, τ is a homeomorphism. Hence the peak $\{*\}$ is an inner point of $V_1 = \tau(U_1)^\cdot$. On the other hand we obtain that for any $f \in A_o$ the function $\hat{f} \circ \tau^{-1}$ is approximable on V_1 by generalized polynomials, i.e. that $\hat{f} \circ \tau^{-1}$ belongs to the algebra $A^\nu(V_1)$. It is clear that the same fact holds for any element of A. Moreover $\hat{f} \circ \tau^{-1}$ can be extended from V_1 up to a big disc $\Delta_G(d)$ as an element of $A^\nu(d)$. In fact, the functions $q_r = p_r \circ \lambda\hat{u} \circ \tau^{-1}$ are defined not only on V_1, but also on $\overline{\Delta}_G(\eta)$ and present the partial sums of a generalized power series, converged on any big disc $\overline{\Delta}_G(d)$ with $0 < d < \max\{\lambda | (\lambda, g) \in V_1 \subset \overline{\Delta}_G(\eta)\}$, according to an appropriate generalized version of Abel's theorem for power series. Using Lemma 1 one can show that the set $V = \Delta_G(d) \cap V_1$, where d is as above, is a generalized-analytic subset of the big disc $\Delta_G(d)$. Taking U =

$\tau^{-1}(V)$ we complete the proof of the theorem.

<u>Theorem 3</u>. Let $\Theta \in$ sp A is such that in $M = \mathrm{Ker}\,\Theta$ there exists a sequence $\{g_{p(j)}\}_{j=1}^{\infty}$, $\|g_{p(j)}\| = 1$, for which:

1) $\{g_{p(j)}\}$ is a multiplicative subsemigroup, isomorphic to the additive semigroup $\mathbb{Q}^{\nu} = \mathbb{Q} \cap [\nu, +\infty) \cup \{0\}$, $0 < \nu < 1$, and

2) the set M_o of elements f of M, presentable uniquely in the form

$$f = \sum_{p(j) \in \mathbb{Q}^{\nu}}{}_{1}^{\infty} f_j\, g_{p(j)}, \sum \|f_j\| < \infty$$

is dense in M.

Then there exist a number $d > 0$, a metric neighbourhood U of Θ, a generalized-analytic set V in some big disc $\Delta_G(d)$, where $G = \widehat{\mathbb{Q}}$, and a homeomorphism $\tau : U \to V$, $\tau(\Theta) = *$, such that $\widehat{f} \circ \tau^{-1}$ is a generalized-analytic function from the algebra $A^{\nu}(V)$ for any $f \in A$.

Indeed, if $f \in M_o$ then $f = \quad f_j\, g_{p(j)}$ with some $f_j \quad A$, such that $\|f\|_1 = \sum \|f_j\| < \infty$. Because for every $j = 1, 2, \ldots$ we have that $f_j - \Theta(f_j) = \sum_k f_{jk}\, g_{p(k)}$ for some f_{jk} with $\sum_k \|f_{jk}\| < \infty$, then $f = \sum \Theta(f_j)\, g_{p(j)} + \sum_{j,k} f_{jk}\, g_{p(k)}\, g_{p(j)}$, where $\sum |\Theta(f_j)| = \sum \|f_j\| < \infty$, so that the kernel M satisfies the conditions of Theorem 2.

<u>Corollary 4</u>. The set U in Theorem 3 can be choosen open in the weak*-topology.

If $A_o = M_o + \mathbb{C}$, it is sufficient to show that every $(A_o, \|\ \|_o)^*$-metric neighbourhood of Θ contains a set of type $\{\varphi \in \mathrm{sp}\,A \mid |\varphi(g_1)| < \varepsilon\}$ for some $\varepsilon > 0$. In fact, if $S = \{f \in A_o \mid f = \sum f_j\, g_{p(j)},\ \|f\|_1 \leq 1\}$, then $M_o = \bigcup_m m.S = \bigcup_m [m.S]_o$. Applying the Bair category theorem, we can see that $[S]_o$ contains a $\|\ \|_1$-neighbourhood of 0, say $\{f \in M_o \mid \|f\|_1 \leq 4\sigma\}$. It is easy to see inductively that $B_{M_o} \subset \frac{1}{2\sigma} S$. Consequently for any $f \in A_o$ there exist elements $f_j \in A$, such that $\sum \|f_j\| \leq \frac{1}{2\sigma}\|f\|_o$ and for which $f = \sum_1^{\infty} f_j\, g_{p(j)}$. If now $\|f\|_o \leq 1$, $f \in A_o$, then $|\varphi(f) - \Theta(f)| = \max|\varphi(g_{p(j)})| \sum|\varphi(f_j)| \leq \max|\varphi(g_1)|^{p(j)} \cdot \sum\|f_j\| \leq \frac{1}{\sigma}|\varphi(g_1)|$. Now the metric neighbourhood $\{\varphi \in \text{sp } A \mid \|\varphi - \Theta\|^{0} < \varepsilon\}$ of Θ contains the weak*-neighbourhood $\{\varphi \in \text{sp } A \mid |\varphi(g_1)| < d = (\varepsilon\sigma)^{1/\nu}\}$, Q. E. D.

The functionals φ_{*} = "the evaluation on $*$" of algebras $A^{\nu}(\varepsilon)$ are typical examples of functionals, satisfying Theorems 2 and 3. Completely different is the situation of the algebra $A_{\hat{\mathbb{Q}}}(\varepsilon)$; because it do not admit nontrivial derivations at φ_{*}. This is not surprising for the part of the point φ_{*} in sp $A_{\hat{\mathbb{Q}}}(\varepsilon)$ coincides with φ_{*} itself.

Note, that the right inequality for ν in Theorems 2 and 3 can be dropped without loss of generality. If the algebra A_0 coincides with A, then the proofs of Theorems 2 and 3 hold for the usual norm of A^{*} and in this case the set $U \ni \Theta$ is a neighbourhood of Θ in the metric topology, inherited by sp A from A^{*}. An independent proof of Theorem 3 for the case of $A_0 = A$ is given in [6]. The present proof, however, gives a little more, namely, that the point $*$ is nonisolated in the topology of A^{*}.

References

1. A. Browder. Point derivations and analytic structure in the spectrum of a Banach algebra. J. Funct. Anal. 7(1971), 156-164.
2. A. Gleason. Finitely generated ideals in Banach algebras. J. Math. mech. 13(1964), 125-132.
3. T. Gamelin. Uniform algebras, Prentice-Hall, Englewood Cliffs, N. J., 1969.
4. T. Tonev. Infinite dimensional analytic sets in maximal ideal space. Math. and Educ. in Math., Sofia (1978), 510-514.
5. R. Arens, I. Singer. Generalized analytic functions. Trans. Amer. Math. Soc. 81(1956), 379-393.
6. T. Tonev. Generalized analytic structure in the spectrum of a uniform algebra. Comt. rend. de l'Acad. bulgare des Sci. 31(1978), 799-802.

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