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ON THE CONVERGENCE OF SOME QUADRATURE PROCESSES

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1. Introduction. Let S be a subset of realvalued functions defined on the unit interval $I=[0,1]$. Furthemore, we take the following subsets of functions on I: R (Riemann-integrable), V (boundid variation), C (continuity), H_ω (with modulus of continuity does not exceed the given modulus of continuity $\omega(.)$) and L_p (with norm $\|f\|_p=\{\int_0^1 |f(t)|^p dt\}^{1/p}$, $(1 \le p \le \infty)$. We denote by $W^rS, r=0,1,\dots$ the set of functions f such that (r-1) st derivative $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in S$; for $r=0, W^0S = S$. In the sequel, we shall regard pair infinte triangular matrices (X,P) : the knot matrix $X=\{x_k^{(n)}\}$ and weight matrix $P=\{p_k^{(n)}\}$, where $p_k^{(n)} \ge 0, k=1,\dots,n$, $\sum_{k=1}^n p_k^{(n)} =1, n=1,2,\dots$ For the given pair (X,P) and $n=1,2,\dots$ let us denote ([1], [2]) the L_p-discrepancy $(1\le p\le \infty)$ by

$$D_n^{(p)}(X,P) = \|D_n(X,P,.)\|_p \text{ ,where}$$

$$D_n(X,P,t)=\Big|t- \sum_{x_k^{(n)} \le t} p_k^{(n)}\Big| \quad \text{and } \{x\}=x-[x].$$

The present paper deals with the quadrature process based on the pair (X,P) $(r\ge 0, f \in W^rR)$:

$$\int_0^1 f(t)dt = \sum_{k=1}^n \sum_{j=0}^r A_{kj}^{(n)} f^{(j)}(x_k^{(n)}) + R_n^r(f;X,P)$$

where $$A_{kj}^{(n)} = \frac{(a_k^{(n)}-x_k^{(n)})^{j+1} - (a_{k-1}^{(n)} - x_k^{(n)})^{j+1}}{(j+1)!},$$

$$a_0^{(n)}=0,\ a_k^{(n)}= \sum_{i=1}^k p_i^{(n)} \text{ ,} j=0,\dots,r; k=1,\dots,n; n=1,2,\dots$$

We note, that in the simple case r=0, we have the general quadrature process($f\in R$): $\int_0^1 f(t)dt = \sum_{k=1}^n p_k^{(n)} f(x_k^{(n)}) + R_n^o(f;X,P)$

Usually, we set $R_n^r(f)$, $D_n^{(p)}$ and D_n instead of $R_n^r(f;X,P)$, $D_n^{(p)}(X,P)$ and $D_n^{(\infty)}(X,P)$ respectively. The special case $p_k^{(n)} = 1/n$, $k=1,\dots,n$ and $f\in W^rC$ ($p=\infty$) was studied earlier :

(1) $\left|R_n^o(f)\right| \le \omega(f;D_n)$, (r=0, by H.Niederreiter [3])

(2) $\left|R_n^r(f)\right| = \frac{D_n^r}{(r-1)!} \int_0^1 (1-t)^{r-1}\omega(f^{(r)};tD_n)dt$ ($r\ge 1$, in [4])

In the case $f\in W^rR$, $r\ge 1$ and $p=\infty$, V.Hristov [5] proved the inequality

(3) $\left|R_n^r(f)\right| = \frac{D_n^r}{(r-1)!} \tau(f^{(r)}; 4D_n)_L$,

where $\tau(f;t)_p = \|\omega(f,.;t)\|_p$ ($\omega(f,x;t) = \sup\{|f(x')-f(x'')| : x',x''\in[x-t/2,x+t/2]\cap I\}$) is the average modulus of smoothness (see Bl.Sendov, V.Popov [6]). In case r=0 and $f\in R$, results of type (3) are due to P.Proinov [2](on I)and to one of authors [7] (on the s-dimensional unit cube $[0,1]^s$) .

In connection with the estimation (1) -(3) , we mark the problem of pair matrices (X,P) with $\lim D_n = 0$, $n\to\infty$ (so calleduniformly distributed matrix X with weight P)is studied by many authors (see, example [1]). It is known [1] that for some pairs (X,P) ($n\to\infty$)

(4) $D_n^{(2)} = O(\sqrt{\ln n/n})$, $D_n = O(\ln n/n)$ and $D_n^{(2)} \ll D_n$.

In the next section we shall formulate some estimations of R_n^r better than inequalities (1) - (3) .

2 Main Results.

Everywhere in this section we assume that (X,P) is given a pair of infinite triangular matrices, $p\in[1,\infty]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1 : If $f\in R$, then for n=1,2,...

$\left|R_n^o(f;X,P)\right| \le 2\,\tau(f;2D_n^{(q)})_p$. In case $f\in C$, we have

$\left|R_n^o(f;X;P)\right| = 2\,\omega(f,D_n^{(1)})$.

Immedialy, from Theorem 1 and the properties of moduli τ and ω [6] we obtain

Corollary 1 : For n=1,2,...

(i) $f \in W^1 R \quad \Rightarrow \left| R_n^o(f) \right| \le 4\, D_n^{(q)} \left\| f' \right\|_p ;$

(ii) $f \in W^1 C \quad \Rightarrow \left| R_n^o(f) \right| \le 2\, D_n^{(1)} \left\| f' \right\|_C$

(iii) $f \in H^\alpha,\ 0<\alpha \le 1 \quad \Rightarrow \left| R_n^o(f) \right| \le 2\, (D_n^{(1)})^\alpha .$

We note an application of Theorem 1 to the estimation of exponential sums .

<u>Corollary 2</u> .Let $x_1, x_2, \dots, x_n$ be real numbers with discrepancy $D_n^{(q)}$ about weights $\{p_k\}_{k=1}^n$, $p_k \ge 0$, $\sum_{k=1}^n p_k = 1$:Then ($p \in [1,\infty]$, $\frac{1}{p}+\frac{1}{q}=1$

$$L \equiv \left| \sum_{k=1}^n p_k \exp(2\pi i x_k) \right| \le 8\, (\pi)^{1-1/(2p)} D_n^q \left\{ \Gamma\left(\frac{p+1}{2}\right) / \Gamma\left(\frac{p+2}{2}\right) \right\}^{1/p}$$

In case $q=1$, $L \le 4\pi\, D_n^{(1)}$.

<u>Remark 1</u> : It is known [1] that $L \le 4\, D_n^{(\infty)}$ .For some pair of matrices (X,P),the established estimation in Theorem 1 and corollaries 1 , 2 are better for the order than (1) and these in [1] , [2] , in accordance with inequalities $D_n^{(\infty)} < D_n^{(1)} \le D_n^{(2)}$ and (4) .

<u>Remark 2</u> : In an arbitrary number s, $s \ge 2$ of dimennsions, an upper bound for the integration error was obtained in [7] of the form

$$(2^s + 1)\, \tau(f; 2\, [D_n^{(q)}]^{1/s})_p .$$

For the integer $r=1,2,\dots$; $n=1,2,\dots$,let us introduce the numbers

$$C_n(X,P,r,p) = \max\left\{ \frac{D_n^r}{(rp+1)^{1/p}} ,\ \left[\frac{D_n^{rp} + (D_n - H)^{rp}}{2} \right]^{1/p} \right\} \quad \text{and}$$

$C_n(X,P,r) = C_n(X,P,r,1)$, where $H = \max(h_1, h_2)$,

$$h_1 = \min\left\{ x_k^{(n)} - x_{k-1}^{(n)} : x_k^{(n)} \ne x_{k-1}^{(n)}, k=1,\dots,n \right\},\ h_2 = \min\left\{ p_k^{(n)} : p_k^{(n)} \ne 0, k=1,\dots n \right\} .$$

<u>Theorem 2</u> : If $f \in W^r R$ $(r \ge 1)$, $n=1,2,\dots$,and L_n is a function defined on I wich ($k=1,\dots n$)

$$L_n(f;X,P;t) = \sum_{j=0}^r \frac{f^{(j)}(x_k^{(n)})}{j!} (t - x_k^{(n)}) \quad \text{for} \quad t \in [a_{k-1}^{(n)}, a_k^{(n)}]$$

then $\left\| f - L_n \right\|_p \le \frac{C_n(X,P,r,p)}{r!}\, \tau(f^{(r)}; 2\, D_n)_p$:If,in addition,

(5) $\quad x_k^{(n)} \in \left[a_{k-1}^{(n)}, a_k^{(n)}\right] \quad ,k=1,\dots,n$

then $\quad \|f-L_n\|_p \le \dfrac{D_n^r}{r!(rp+1)^{1/p}} \; \tau\,(f^{(r)};2\,D_n)_p .$

From Theorem 2 it follows in particular (p=1)

$$\left|R_n^r(f)\right| \le \frac{C_n(X,P,r)}{r!}\,\tau(f^{(r)};2\,D_n)_L \text{ ,and if (5) ,then}$$

$$\left|R_n^r(f)\right| \le \frac{D_n^r}{(r+1)!}\,\tau\,(f^{(r)};2\,D_n)_L .$$

Theorem 3 : For all $f\in C^r(r\ge 1)$,

$$\left|R_n^r(f;X,P)\right| \le \frac{C_n(X,P,r)}{(r-1)!}\int_0^1 (1-t)^{r-1}\omega(f^{(r)};tD_n)dt. \text{ In case (5)}$$

$$\left|R_n^r(f;X,P)\right| \le \frac{r}{(r+1)!}\,D_n^r\int_0^1 (1-t)^{r-1}\omega(f^{(r)};tD_n)dt .$$

3. Proof of the Main Results. The proof of Theorem 1 is based on the method used in [7] .It is not hard to establish

Lemma 1 : Let we have a set of knots $0\le x_1\le x_2\le\dots\le x_n\le 1$ and weights $p_k\ge 0,k=1,\dots,n$ with $a_k=\sum_{i=1}^k p_i$, $a_n=1$. Then ,the uniform discrepancy $D_n^{(\infty)}=\max\left\{\left|x_k-a_k\right|,\left|x_k-a_{k-1}\right|:k=1,\dots,n\right\}$ is not changed in case of elimination of zero weight or/and of dublicated knot .

Lemma 1 enables us to assume,without loss of generality,that $\left\{x_k^{(n)}\right\}\ne\left\{x_i^{(n)}\right\}$ $(k\ne i)$ and $p_k^{(n)}\ne 0$ in the following considerations.

Proof of Theorem 2 : Consequtively,we get :

$$(6)\quad \left\|f(.)-L_n(f,X,P;.)\right\|_p=\left\{\sum_{k=1}^n\int_{a_{k-1}^{(n)}}^{a_k^{(n)}}\left|f(t)-L_n(f;X,P;t)\right|^p dt\right\}^{1/p}$$

$$\le\left\{\sum_{k=1}^n\int_{a_{k-1}^{(n)}}^{a_k^{(n)}}\left[\frac{\left|t-\{x_k^{(n)}\}\right|^r}{(r-1)!}\int_0^1(1-z)^{r-1}\left|f^{(r)}\left[\{x_k^{(n)}\}+(t-\{x_k^{(n)}\})z\right]-\right.\right.\right.$$

$$-f^{(r)}(\{x_k^{(n)}\})\,dz]^p dt\}^{1/p} = \left\{\sum_{k=1}^{n} M_{k,r}\right\}^{1/p} .$$

We consider the cases 1) $\{x_k^{(n)}\} < a_{k-1}^{(n)}$,2) $\{x_k^{(n)}\} > a_k^{(n)}$ and 3) $x_k^{(n)} \in [a_{k-1}^{(n)}, a_k^{(n)}]$.

In cases 1) and 2) ,for arbitrary $t,u \in [a_{k-1}^{(n)}, a_k^{(n)}]$:

$$\left| f^{(r)}\left[\{x_k^{(n)}\} + (t-\{x_k^{(n)}\})z\right] - f^{(r)}(\{x_k^{(n)}\})\right| \le \omega(f^{(r)},u;2\,D_n) \text{ and}$$

$$M_{k,r} = \int_{a_{k-1}^{(n)}}^{a_k^{(n)}} \left[\frac{|t-\{x_k^{(n)}\}|^r}{(r-1)!} \int_0^1 (1-z)^{r-1}\,\omega(f^{(r)},u;2\,D_n)dz\right]^p dt \le$$

$$\le \left(\frac{1}{r!}\right)^p \frac{D_n^{rp} + (D_n - H)^{rp}}{2}\left[\omega(f^{(r)},u;2\,D_n)\right]^p du \text{ . Hence}$$

$$M_{k,r} \le \left(\frac{1}{r!}\right)^p \frac{D_n^{rp} + (D_n-H)^{rp}}{2} \int_{a_{k-1}^{(n)}}^{a_k^{(n)}} \left[\omega(f^{(r)},u;2\,D_n)\right]^p du .$$

In case 3) we get

$$M_{k,r} \le \left(\frac{1}{r!}\right)^p \frac{D_n^{rp}}{rp+1} \int_{a_{k-1}^{(n)}}^{a_k^{(n)}} \left[\omega(f^{(r)},u;2\,D_n)\right]^p du$$

The last inequalities and (6) complete the proof of Theorem 2 .

<u>Proof of Theorem 3</u> :For $f \in C^r (r \ge 1)$ and p=1,using (6) we have

$$R_n^r(f;X,P) \le$$

$$\le \frac{1}{(r-1)!} \sum_{k=1}^{n} \int_{a_{k-1}^{(n)}}^{a_k^{(n)}} \left[\left|t-\{x_k^{(n)}\}\right|^r \int_0^1 (1-z)^{r-1}\,\omega(f^{(r)};|t-\{x_k^{(n)}\}|)dz\right]dt \le$$

$$\le \frac{1}{(r-1)!} \int_0^1 (1-z)^{r-1}\,\omega(f^{(r)};zd_n)dz \sum_{k=1}^{n} \int_{a_{k-1}^{(n)}}^{a_k^{(n)}} \left|t-x_k^{(n)}\right|^r dt$$

Now,as in the proof of Theorem 2 we get

$$\int_{a_{k-1}^{(n)}}^{a_k^{(n)}} |t-\{x_k^{(n)}\}|^{rp} dt = \left[c_n(X,P,r,p)\right]^p (a_k^{(n)} - a_{k-1}^{(n)})$$

which comlete the proof of Theorem 3 .

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