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LOCAL INTERPOLATION SPLINES OF ODD DEGREE

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1. Introduction and Results. In the present paper we study the approximation of continous function in R^s (s=1,2) with the so called local interpolation splines (L.I.S.) in C-metric. Introduced in ([1] , [2]), the L.I.S. in natural way generalize the constructions of polygons ([3], [4]), of $\wp$-splines ([5], [6]), of k-splines [7] and of finite elements ([8], [9]).

We begin with introduction of some notations. For the function $f \in C^k[0,1]$ ($k \geq 0$) and $x, x' \in [0,1]$ to set

$$S_k(f, x'; x) = \sum_{i=1}^{k} \frac{f^{(i)}(x')}{i!} (x - x')^i$$

For a given modulus of continuity ω and $k \geq 1$, we denote by $W^k H_\omega$ the set of function f defined on [0,1] such that (k-1) derivative $f^{(k-1)}$ is absolutely continous and the modulus of continuity of $f^{(k)}$ does not exceed ω .

Let $\tilde{P}^1(h)$ ($h > 0$) be a class of functions P(.) with the following properties: 1) $P \in C[0,h]$, 2) $P(u) \geq 0, u \in [0,h]$, 3) $P(u) + P(h-u) = 1, u \in [0,h]$, 4) $P(u) \geq P(h-u), u \in [0,h/2]$.

Let be given a partition of [0,1] $\Delta: 0 = x_0 < x_1 < \dots < x_n = 1$ with $h_i = x_i - x_{i-1}$, $\bar{\Delta} = \max h_i$, $i = 1,\dots,n$.

The set of local interpolation splines $Lis(k, \tilde{P}^1, \Delta)$ is introduced as a collection of all linear operators $L(., \Delta; x) : C^k[0,1] \to C[0,1]$ defined for $x \in [x_{i-1}, x_i]$, $i = 1,\dots,n$ by

$L(f,\Delta;x)=S_k(f,x_{i-1};x)P_i(x-x_{i-1})+S_k(f,x_i;x)P_i(x_i-x)$ where $P_i\in\widetilde{P}^1(h_i), i=1,\dots,n$

In the next Theorem we deal with the estimation of

$$E_k(W^kH_\omega;\Delta,\widetilde{P}^1)=\inf_{L\in Lis(k,\widetilde{P}^1,\Delta)}\ \sup_{f\in W^kH_\omega}\|f(.)-L(f,\Delta;.)\|_{C[0,1]}$$

for odd $k\geq 1$;moreover, we find an extremal operator $L_*\in Lis(k,\widetilde{P}^1,\Delta)$ for which

$$\sup_{\omega(t)\neq 0} E_k(W^kH_\omega;\Delta,\widetilde{P}^1)=\sup_{\omega(t)\neq 0}\ \sup_{f\in W^kH_\omega}\|f(.)-L_*(f,\Delta;.)\|_{C[0,1]}$$

<u>Theorem 1</u> : For $k=1,3,5,\dots$

$$\sup_{\omega(t)\neq 0}\frac{E_k(W^kH_\omega;\Delta,\widetilde{P}^1)}{\int_0^{\bar{\Delta}} t^{k-1}\omega(t)dt}=\frac{1}{2^{k+1}(k-1)!}$$

The extremal operator L is based on the functions

$$P_i^*(u)=\frac{(h_i-u)^k}{u^k+(h_i-u)^k},\ i=1,\dots,n. \tag{1}$$

Let $\widetilde{P}^2(h,q)$ $(h,q>0)$ be a set of functions P defined on $\Omega=[0,h]\times[0,q]$ with properties : 1) $P\in C(\Omega)$; 2) $P(u,v)\geq 0$, $(u,v)\in\Omega$; 3) $P_1+P_2+P_3+P_4=1$, where $P_1=P(u,v)$, $P_2=P(h-u,v)$, $P_3=P(h-u,q-v)$, $P_4=P(u,q-v)$; 4) $P_1+P_4\geq P_2+P_3$, $P_1+P_2\geq P_3+P_4$, $P_1=\max(P_1,P_2,P_3,P_4)$ for $(u,v)\in\Omega_{1/2}=[0,h/2]\times[0,q/2]$; 5) $P(0,0)=1$, $P(h,0)=P(0,q)=P(h,q)=0$.

Let be given a partition $\Delta=\{\Delta_i=[a_i,b_i]\times[c_i,d_i]\}_{i=1}^n$ of the unit square $G^2=[0,1]^2$ with $\bigcup_{i=1}^n\Delta_i=G^2$, $Int(\Delta_i\cap\Delta_j)=\emptyset$, $i\neq j$; $h_i=b_i-a_i$, $q_i=d_i-c_i$, $i=1,\dots,n$ and $\bar{\Delta}=\max\{\sqrt{h_i^2+q_i^2} : i=1,\dots,n\}$.

We would say, that operator $L(.,\Delta;x,y)$: $C(G^2)\to C(G^2)$ belongs to $Lis(\widetilde{P}^2;\Delta)$, if it is defined for $(x,y)\in\Delta_i, i=1,\dots,n$ by

$$L(f,\Delta;x,y)=P(u,v)f(a_i,c_i)+P(h_i-u,v)f(b_i,c_i)+P(h_i-u,q_i-v)f(b_i,d_i)+P(u,q_i-v)f(a_i,d_i), \tag{2}$$

where $u=x-a_i$, $v=y-c_i$ and $P\in \widetilde{P}^2(h_i,q_i)$.

For a given modulus of continuity ω, let us introduce the class H^2_ω of two-dimensional functios defined on G^2 :

$$H^2_\omega = \{ f: |f(x,y)-f(x'-y')| \le \omega(\sqrt{(x-x')^2+(y-y')^2} , (x,y),(x',y') \in G^2 \}.$$

Theorem 2 : For every partition Δ of G^2 and operator $L\in Lis(\widetilde{P}^2,\Delta)$

$$\sup_{f\in H^2_\omega} \|f(.) - L(f, \quad ;.,.)\|_{C(G^2)} \le 3\omega(\bar{\Delta}/4) .$$

The constant 3 cannot be made smaller in the general case (arbitrary ω and Δ), a.e. for some partition Δ_*, modulus ω_*, function f_* and operator L_* we have

$$\|f_*(.,.) - L_*(f_*, \Delta_*;.,.)\|_C = 3\,\omega_*(\bar{\Delta}_*/4).$$

Remark : The properties of functional classes $\widetilde{P}^1$ (1) - 4)) and $\widetilde{P}^2$ (1) - 5)) are consequence of the local approximation characteristics of the operators L , naturally desired in this case (see Theorem 1 [2] and Lemma B [1]) .

2 . Proof of Theorems .

Proof of Theorem 1 : For every $f\in W^kH_\omega$, $L\in Lis(k,\widetilde{P}^1, \Delta)$ and $x\in[x_{i-1},x_i]$, $i = 1,\dots,n$ we have

$$(3)\quad f(x)-L(f,\Delta;x) = \frac{P_i(x-x_{i-1})}{(k-1)!}\int_{x_{i-1}}^{x} (x-t)^{k-1}f^k(t)dt \; +$$

$$+ \left(\frac{x-x_{i-1}}{x-x_i}\right)^k \frac{P_i(x_i-x)}{(k-1)!} \int_{x_{i-1}}^{x} (x-t)^{k-1}f^{(k)}\left[x-(x-t)\frac{x-x_i}{x-x_{i-1}}\right]dt$$

For an arbitrary function $f_1\in W^kH_\omega$ with $f_1^{(k)}(t) = t$ and

$$\bar{x}_i = (x_i+x_{i-1})/2 \;,\; f_1(\bar{x}_i)-L(f_1,\Delta;\bar{x}_i)= \frac{1}{2(k-1)!}\int_{x_{i-1}}^{\bar{x}_i} (\bar{x}_i-t)^{k-1}\times$$

$$\times (2t - 2\bar{x}_i)dt = -\frac{1}{2(k-1)!}\int_{x_{i-1}}^{\bar{x}_i} 2(\bar{x}_i-t)^{k-1}(\bar{x}_i-t)dt =$$

$$= -\frac{1}{2^{k+1}(k-1)!}\int_0^{h_i} t^{k-1}\omega_1(t)dt \quad ,\text{where } \omega_1(t) = t \;.\; \text{Hence,}$$

$$(4)\quad \sup_{\omega(t)\neq 0}\ \sup_{f\in W^kH_\omega} \|f(.) - L(f, \quad ;.)\|_C \Big/ \int_0^{\bar{\Delta}} t^{k-1}\,\omega(t)dt \ge 1.$$

On the other hand, for the operator L_* based on the functions $P_i^*, i=1,\dots,n$ (see (1),(3)), we have ($x\in[x_{i-1},x_i]$)

$$\left|f(x) - L_*(f,\Delta;x)\right| \leq$$

$$= \frac{1}{(k-1)!}\cdot\frac{(x_i - x)^k}{(x-x_{i-1})^k+(x_i-x)^k}\int_{x_{i-1}}^{x}(x-t)^{k-1}\omega\left(f^{(k)};(x-t)\frac{h_i}{x-x_{i-1}}\right)dt$$

$$\leq \frac{1}{(k-1)!\,2^{k+1}}\int_0^{\bar\Delta} t^{k-1}\omega(t)dt, \quad \text{therefore}$$

$$\left\|f(\cdot) - L_*(f,\Delta;\cdot)\right\|_C \leq \frac{1}{(k-1)!2^{k+1}}\int_0^{\bar\Delta} t^{k-1}\omega(t)dt.$$

The last inequality and (4) complete the proof of Theorem 1 .

Proof of Theorem 2 : It is enough to regard the function f and the operator L on $\Omega = [0,h]\times[0,q]$ with $d = \sqrt{h^2+q^2}$ ($h,q > 0$). From the representation (2), for every point $(u,v)\in\Omega$ we obtain

$$\left|f(u,v) - L(f,\Delta;u,v)\right| \leq \sum_{i=1}^{4} P_i\,\omega(d_i) \equiv S(u,v)$$

where $d_1=\sqrt{u^2+v^2}$, $d_2=\sqrt{(h-u)^2+v^2}$, $d_3=\sqrt{(h-u)^2+(q-v)^2}$, $d_4=\sqrt{u^2+(q-v)^2}$.

Now, we shall establish the inequality

(5) $$S(u,v) \leq 3\,\omega(d/4) \quad \text{for } (u,v)\in\Omega .$$

Due to the symmetry about point $(h/2,q/2)$, it is enough to prove (5) for $(u,v)\in\Omega_{1/2} = [0,h/2]\times[0,q/2]$. It can be shown that in this case the inequalities

(6) $d_1\leq d/2$, $\max(d_2,d_4) \geq d/2$, $d_3\geq d/2$ and $\min(d_2,d_4) \leq 3d/4$ are fulfilled, Hence, for the points $(u,v)\in\Omega_{1/2}$ are possible the following cases : A) $d_i\leq 3d/4$, $i=2,3,4$; B) $\min(d_2,d_4) \leq d/2$; C) $d/2 < \min(d_2,d_4) \leq 3d/4$, $d_3\leq 3d/4$; D) $d/2 < \min(d_2,d_4) \leq \max(d_2,d_4) \leq 3d/4$; E) $d_3 > 3d/4$, $d/2 < \min(d_2,d_4) \leq 3d/4$, $\max(d_2,d_4) > 3d/4$.

Note, there is not any point (u,v) satisfying case E) with $d/4 < d_1 \leq d/2$.

Now, from (6) and properties 1) - 5) of P_i for every case A) - E) it is not hard to establish inequality (5).

The constant 3 is exact.. Let us regard the set $\Omega_* = [0,h]\times[0,h]$ with $h = d/\sqrt{2}$. We define the function $P_* \in \tilde{P}^2(h,h)$:

$$P_*(u,v) = \begin{cases} -3u/(2h) + 1 , & h/2 \geq u \geq v \geq 0 \\ -3v/(2h) + 1 , & h/2 \geq v \geq u \geq 0 \end{cases} ,$$

$$P_*(h-u,v) = P_*(u,h-v) = P_*(h-u,h-v) = \frac{1- P_*(u,v)}{3} ,$$

$(u,v) \in \Omega_{1/2} = [0,h/2]\times[0,h/2]$.

For $0 < \varepsilon < \frac{d(\sqrt{10} -3)}{4}$,we consider the modulus of continuity $(k = 1,2,\dots)$

$$\omega_*(t) = \begin{cases} \frac{d}{4} (k + \frac{4t - kd}{4\varepsilon}) , & t \in \left[\frac{kd}{4} , \frac{kd}{4} + \varepsilon\right] , \\ \frac{d}{4} (k + 1) , & t \in \left[\frac{kd}{4} + \varepsilon , (k + 1) \frac{d}{4}\right] . \end{cases}$$

Finaly we take the function $f_*(u,v) = \omega_*(\sqrt{(u - h/2)^2 + v^2})$ and the operator $L_* \in \mathrm{Lis}(\tilde{P}^2, \Delta)$ $(\Omega_* \in \Delta)$ based on P_* and receive :

(7) $\left| f_*(h/2,0) - L_*(f_*, \Delta;h/2,0)\right| = \left| L_*(f_*, \Delta; h/2 ,0)\right| =$

$= \left| P_*(h/2,0)f_*(0,0) + P_*(h/2,0)f_*(h,0) + P_*(h/2,h)f_*(h,h) + \right.$
$\left. + P_*(h/2,h)f_*(0,h) \right| =$
$=1/2 \left[\omega_*(h/2) + \omega_*(\sqrt{5/4}\, h)\right.$.

On the other hand, the following inequalities are hold :

$d/2 > h/2 = d/(2\sqrt{2}) > d/4 + \varepsilon$, $d > \sqrt{5/4}\, h = d\sqrt{5}/(2\sqrt{2}) > 3d/4 + \varepsilon$.

Hence ,we get

$$\left| f_*(h/2,0) - L_*(f_*, \Delta; h/2,0)\right| = 1/2 \ (d/2 + d) = 3\,\omega_*(d/4)$$

which complete the proof of Theorem 2 .

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