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ESTIMATES OF THE APPROXIMATION OF DERIVATIVES WITH THE HELP OF SMOOTHING SPLINES

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1. Introduction. Let $\Delta : a = x_0 < \ldots < x_N = b$ be the mesh on the interval $[a, b]$ and suppose that we know the values of the function $f \in W_2^2(a, b)$ in the knots of the mesh Δ with some error

$$|f(x_i) - z_i^0| \leq \varepsilon_i , \quad \varepsilon_i \geq 0, \quad i = 0, \ldots , N.$$

We denote by C the set of functions $g \in W_2^2(a, b)$ such that $|g(x_i) - z_i^0| \leq \varepsilon_i , \quad i = 0, \ldots , N$. The following problems are known in the theory of splines:

1. To find a function $S_1(x) \in C$, such that

$$\int_a^b |S_1''(x)|^2 dx = \min_{g \in C} \int_a^b |g''(x)|^2 dx .$$

2. To find a function $S_2(x)$ such that

$$\int_a^b |S_2''(x)|^2 dx + \sum_{i=0}^N \rho_i^{-1} (S_2(x) - z_i^0)^2 =$$
$$= \min_{g \in W_2^2(a,b)} \left(\int_a^b |g''(x)|^2 dx + \sum_{i=0}^N \rho_i^{-1} (g(x_i) - z_i^0)^2 \right),$$

where $\rho_i \geq 0, \ i = 0, \ldots , N.$

It is well known that the solutions of the problems 1 and 2 exist and are the natural cubic splines satisfying some conditions of characterization [1]. Splines, satisfying the boundary conditions of the first type (with the fixed first derivatives on the ends) or periodic, minimize the functional of problems 1 and 2 on the corresponding classes of functions. The solution of problem 1 is called by the spline in the convex set, given by the system of inequalities, and the solution of problem 2 is called by the smoothing spline. Since both of them are used in order to liquidate the oscillations arising in the interpolation of the data given with some error we can call both of them by smoothing splines. It is known that the solution of the problem 2 is unique and if some conditions for the class C are satisfied, then the solution of problem 1 is also unique. Let $\bar{h}$ be the maximum of $h_i = x_i - x_{i-1}$, $\varepsilon = \max \varepsilon_i$, $i = 0, \dots, N$. First derivatives of $S_1(x)$ converge to $f'(x)$ in the metric of $C[a,b]$ and $S_1''(x)$ converge to $f''(x)$ in the metric of $L_2(a,b)$ [2] if $\bar{h}, \varepsilon \to 0$, and some conditions on the sequence of meshes are satisfied. Numerical experiments show that the derivatives of the splines S_1 and S_2 approximate the derivatives of f perfectly well. So the question about the estimates of the error of this approximation arises. Some estimates for the first derivatives are suggested in this report. For details see papers [3] and [4].

2. Estimates for the splines in the convex set. Let us suppose that $f(x) \in W^2_\infty(a,b)$. We estimate $|f(x) - S_1(x)|$ in several ways, first by analogy with [2]:

$$|f(x) - S_1(x)| \le 2\varepsilon + \frac{\bar{h}^2}{8} \|f'\|_\infty + \frac{\sqrt{3}}{8} \bar{h}^2 \underline{h}^{-1/2} K, \quad (1)$$

where $\underline{h} = \min_i h_i$, $\quad K = \left(\int_a^b (f''(x))^2 dx\right)^{1/2}$.

Using the interpolating spline $\tilde{S}$ of the function $f(x)$, which satisfies the natural boundary conditions, we have

$$|f(x) - S_1(x)| \le |f(x) - \tilde{S}(x)| + |\tilde{S}(x) - S_1(x)|.$$

We obtain the following estimate by the methods of § 3 of Chapter III of book [5]:

$$\|f - \tilde{S}\|_C \le \frac{17}{48} \bar{h}^2 \|f''\|_\infty .$$

From the system for the first derivatives of the spline we produce the estimate

$$|\tilde{m}_i - m_i| \le \frac{12\varepsilon}{\underline{h}},$$

where $\tilde{m}_i = \tilde{S}'(x_i)$, $m_i = S'(x_i)$. As a consequence of this estimate, we have

$$|\tilde{S}(x)-S_1(x)| \leq \frac{5\varepsilon}{2} + \frac{3\varepsilon \bar{h}}{\underline{h}} .$$

As a result, we obtain the estimate

$$\|f(x)-S_1(x)\|_C \leq \frac{17}{48}\bar{h}^2\|f''\|_\infty + \frac{5\varepsilon}{2} + \frac{3\varepsilon\bar{h}}{\underline{h}} . \tag{2}$$

If the sequence of the meshes satisfies the inequality

$$\rho = \max_{|i-j|=1} \frac{h_i}{h_j} < (1+\sqrt{13})/2 ,$$

then we can use B-splines and from the system cited on page 141 in [5] we have

$$\|\tilde{S}(x)-S_1(x)\|_C \leq 2\varepsilon \cdot D ,$$

where $D = \dfrac{(2+\rho)(1+\rho+\rho^2)}{\rho(3+\rho-\rho^2)}$. And then the estimate follows

$$\|f(x)-S_1(x)\|_C \leq \frac{17}{48}\bar{h}^2\|f''\|_\infty + 2\varepsilon D . \tag{3}$$

Let us denote the right parts of the inequalities (1-3) by γ_ℓ , $\ell = 1, 2, 3$. It is easy to prove [2] that

$$\eta = \|f''(x)-S_1''(x)\|_{L_2(a,b)} \longrightarrow 0 \quad \text{if} \quad \gamma_\ell \rightarrow 0 ,$$

where ℓ is one of the numbers 1, 2, 3. Using the methods of paper [6], we obtain the inequality

$$\|g'\|_{C[a,b]} = \|g'\|_{L_\infty(a,b)} \leq 2\delta^{-1}\|g\|_{L_\infty(a,b)} + \frac{\sqrt{2}}{2}\delta^{1/2}\|g''\|_{L_2(a,b)},$$

if $g \in W_2^2(a,b)$, and $0 < \delta \leq b-a$. So we have

$$\|f'(x)-S_1'(x)\|_C \leq 2\delta^{-1}\gamma_\ell + \frac{\sqrt{2}}{2}\delta^{1/2}\eta .$$

Let $\delta = Q_\ell \gamma_\ell^{2/3}$, and Q_ℓ be such that $Q_\ell \gamma^{2/3} \leq b-a$, then we finally obtain the estimates

$$\|f'(x)-S_1'(x)\|_C \leq \left(2\varepsilon + \frac{\bar{h}^2}{8}\|f''\|_\infty + \frac{\sqrt{3}}{8}\bar{h}^{-2}\underline{h}^{-1/2}K\right)^{1/3}\left(2Q_1^{-1} + \frac{\sqrt{2}}{2}Q_1^{1/2}\eta\right), \quad (4)$$

$$\|f'(x)-S_1'(x)\|_C \leq \left(\frac{5}{2}\varepsilon + 3\frac{\bar{h}}{\underline{h}}\varepsilon + \frac{17}{48}\bar{h}^2\|f''\|_\infty\right)^{1/3}\left(2Q_2^{-1} + \frac{\sqrt{2}}{2}Q_2^{1/2}\eta\right), \quad (5)$$

$$\|f'(x)-S_1'(x)\|_C \leq \left(2\varepsilon D + \frac{17}{48}\bar{h}^2\|f''\|_\infty\right)^{1/3}\left(2Q_3^{-1} + \frac{\sqrt{2}}{2}Q_3^{1/2}\eta\right). \quad (6)$$

If we suppose that the net is uniform with the step h, then we obtain the estimates

$$\|f'(x)-S_1'(x)\|_C \leq \left(2\varepsilon + \frac{h^2}{8}\|f''\|_\infty + \frac{\sqrt{3}}{8}h^{3/2}K\right)^{1/3}\left(2Q_1^{-1} + \frac{\sqrt{2}}{2}Q_1^{1/2}\eta\right),$$

$$\|f'(x)-S_1'(x)\|_C \leq \left(\frac{9}{2}\varepsilon + \frac{17}{48}h^2\|f''\|_\infty\right)^{1/3}\left(2Q_2^{-1} + \frac{\sqrt{2}}{2}Q_2^{1/2}\eta\right).$$

3. Estimates for the smoothing splines. We shall consider periodic splines on the uniform mesh with the step h. Using the characteristic equations for the spline S_2, we obtain the system of linear equations for the first derivatives m_i of the smoothing spline in the knots of the mesh. Each equation of this system is of the following form:

$$\frac{6\rho_{i-1}}{h^3}\left(1+\frac{24}{h^3}(\rho_i+\rho_{i+1})\right)m_{i-2} + \Big[\Big(1-\frac{24\rho_{i-1}+6\rho_i}{h^3}\Big)\times$$

$$\times\left(1+\frac{24}{h^3}(\rho_i+\rho_{i+1})\right) + \frac{6\rho_i}{h^3}\left(1+\frac{24}{h^3}(\rho_{i-1}+\rho_i)\right)\Big]m_{i-1} +$$

$$+2\Big[\left(1+\frac{24}{h^3}(\rho_i+\rho_{i+1})\right)\cdot\left(1+\frac{9}{h^3}\rho_{i-1}\right) + \left(1+\frac{24}{h^3}(\rho_{i-1}+\rho_i)\right)\left(1+\frac{9}{h^3}\rho_i\right)\Big]m_i +$$

$$+\Big[\Big(1-\frac{24\rho_{i+1}+6\rho_i}{h^3}\Big)\left(1+\frac{24}{h^3}(\rho_i+\rho_{i-1})\right) +$$

$$+\frac{6\rho_i}{h^3}\left(1+\frac{24}{h^3}(\rho_{i+1}+\rho_i)\right)\Big]m_{i+1}+\frac{6\rho_{i+1}}{h^3}\left(1+\frac{24}{h^3}(\rho_{i-1}+\rho_i)\right)m_{i+2}=$$

$$=\frac{3}{h}\Big[(z_i^0-z_{i-1}^0)\left(1+\frac{24}{h^3}(\rho_i+\rho_{i+1})\right)+(z_{i+1}^0-z_i^0)\left(1+\frac{24}{h^3}(\rho_{i-1}+\rho_i)\right)\Big].$$

If $\rho_0=\ldots=\rho_N=\rho$, then the system becomes much more simple:

$$\frac{6\rho}{h^3}m_{i-2}+\left(1-\frac{24}{h^3}\rho\right)m_{i-1}+4\left(1+\frac{9}{h^3}\rho\right)m_i+$$
$$+\left(1-\frac{24}{h^3}\rho\right)m_{i+1}+\frac{6\rho}{h^3}m_{i+2}=\frac{3}{h}(z_{i+1}^0-z_{i-1}^0),\quad i=0,\ldots,N. \tag{7}$$

Matrix of this system has the diagonal dominance if $\rho<h^3/4$ as well as matrix of the analogous system for the moments M_i.

We shall suppose that $f\in W_\infty^5(a,b)$ and let $z'=(f(x_0),\ldots,f'(x_N))$, $m=(m_0,\ldots,m_N)$. Using system (7), we obtain the estimate

$$\|m-z'\|\le\left[\left(\frac{h^4}{30}+6\rho h\right)\|f^V\|_\infty+6\frac{\varepsilon}{h}\right]\Big/\left(4+\frac{24}{h^3}\rho-2\left|1-\frac{24}{h^3}\rho\right|\right), \tag{8}$$

where by the norm of the vector we understand the maximum of modules of its components.

If $h^3/24\le\rho<h^3/4$, then the minimum of the right-hand part achieves by $\rho=h^3/24$ and is equal to

$$1/5\left(\frac{17}{60}h^4\|f^V\|_\infty+\frac{6\varepsilon}{h}\right).$$

If $0\le\rho\le h^3/24$ and the inequality

$$\|f^V\|_\infty<\frac{45\varepsilon}{h^3} \tag{9}$$

is fulfilled, then the minimum of the right-hand part of (8) achieves on the interpolating spline $\rho=0$ and equals

$$\frac{1}{2}\left(\frac{h^4}{30}\|f^V\|_\infty+6\frac{\varepsilon}{h}\right).$$

We note that the value $\rho=h^3/24$ gives the minimum of some estimate and so in concrete cases the minimum of the error of approximation of the first derivatives may be achieved by the other value of ρ. The value $\rho=h^3/24$ may serve only as some reference-point in the choice of parameter of smoothing ρ .

4. Numerical examples. One of the examples examined in [7] is the following: spline S_1 was built by the table of exponential curve e^x rounded to one decimal place, the mesh is uniform with the step $h = 0.1$ on the interval $[0,1]$, $max|S_1(x_i) - e^{x_i}| = 0.03$ while for the interpolating spline this value equals 0.5.

Let us consider $f(x) = 3 \cdot 10^3 Sin(\pi x/20)$ on the interval $[0,40]$, mesh is uniform, $h = 1$. The smoothing splines with parameter ρ are built by the values of the function $f(x)$ rounded up to the first place before the point (Table 1), μ denotes the maximum of the right-hand part of (8), and $\nu = max|S_2'(x_i) - f'(x_i)|$. The results of smoothing of values of function $f(x)$ (on the same mesh) rounded with error equal to 5 are given in Table 2.

Table 1.

ρ	0	1/24	1/12
ν	0,5	0,4	0,6
μ	1,5	0,6	0,8

Table 2

ρ	0	1/24	1/12
ν	5,0	3,7	3,4
μ	15,0	6,0	7,5

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