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OPTIMAL LEBESGUE CONSTANTS FOR POLYNOMIAL INTERPOLATION[(1)]

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1. Introduction

Let $X = \{x_{kn}\}$, $k=1,2,\ldots,n$; $n=1,2,\ldots$, be a fixed triangular matrix with

$$-1 \equiv x_{n+1,n} \le x_{nn} < x_{n-1,n} < \ldots < x_{2n} < x_{1n} \le x_{0n} \equiv 1, \quad n=1,2,\ldots .$$

As it is well-known, in the study of the Lagrange interpolation the behaviour of the Lebesgue functions

$$\lambda_n(X,x) = \sum_{k=1}^{n} |\ell_{kn}(X,x)|$$

and the Lebesgue constants

$$\Lambda_n(X) = \max_{-1 \le x \le 1} \lambda_n(X,x)$$

is of fundamental importance. Here, sometimes omitting the superfluous notations,

[(1)]The detailed version will appear in Acta Math. Hungar.

$$\ell_{kn}(x) = \omega_n(x)[\omega_n'(x_k)(x-x_k)]^{-1}, \quad \omega_n(x) = c_n \prod_{k=1}^{n}(x-x_k).$$

Now we are able to prove the next relation.

If $\gamma = 0.577215\ldots$ is the Euler constant and $\chi := \frac{2}{\pi}(\gamma + \ell n \frac{4}{\pi}) = 0.521251\ldots$, then

$$\Lambda_n^* := \min_X \Lambda_n(X) = \frac{2}{\pi} \ell n\, n + \chi + \sigma(1) \quad .^{(2)}$$

(For the precise formula see (3.3).)

2. Preliminary results

From our point of view, the most important results are as follows (for further references see [12] - [15]).

2.1. In 1914 G. FABER [1], in 1916 S. BERNSTEIN [2] proved that for arbitrary X

$$\Lambda_n(X) > \frac{\ell n\, n}{8\sqrt{\pi}}, \quad n=1,2,\ldots .$$

P. ERDŐS [3] in 1961 obtained that

(2.1) $\frac{2}{\pi} \ell n\, n - c_1 \le \Lambda_n^* \le \frac{2}{\pi} \ell n\, n + c_2$.

(Here and later c, c_1, c_2, ... , const., denote absolute, positive, not necessarily different real numbers.)

(2) The minimum is attained (see Theorem 2.1.).

In 1981 P. ERDŐS and P. VÉRTESI [4] established the Erdős conjecture on the Lebesgue function and proved as follows.

Let $\varepsilon>0$ be any given number. Then for arbitrary matrix X there exist sets H_n with $|H_n|\leq\varepsilon$ and $\eta(\varepsilon)>0$ such that

$$\lambda_n(X,x)>\eta(\varepsilon)\ \ell n\ n \quad \text{if} \quad x\in[-1,1]\setminus H_n \quad \text{and} \quad n\geq n_o(\varepsilon).$$

This result was sharpened by P. VÉRTESI [5] and [6] .

2.2. Another conjecture of P. ERDŐS and S. BERNSTEIN for $\Lambda_n{}^*$ was proved by T.A. KILGORE [7] and C. DEBOOR and A. PINKUS [8] in 1978. To formulate their results (cf. Theorem 2.1.), let us see some observations.

A simple argument shows that for $n\geq 2$ the $\lambda_n(X,x)$ is a piecewise polynomial with $\lambda_n(X,x)\geq 1$ and $\lambda_n(X,x) = 1$ iff $x = x_{kn}$, $1\leq k\leq n$. Between consecutive nodes, $\lambda_n(X,x)$ has a single maximum, in $(-1,x_{nn})$ and $(x_{1n},1)$ it is convex and monotone (see e.g. F.W. LUTTMANN and T.J. RIVLIN [9]).

Let us denote the local maximums by

$$(2.2)\quad \mu_{kn}(X) := \max_{x_{kn}\leq x\leq x_{k-1,n}} \lambda_n(X,x) \ , \ k=1,2,\dots,n+1;\ n\geq 3 \ ,$$

Another simple observation is that to obtain $\Lambda_n{}^*$, "without loss of generality we (can) restrict our attention to those nodal configurations where $-1\equiv x_{nn}$ and $1\equiv x_{1n}$." (see KILGORE [7] , p. 274).

We call these X canonical matrix.

Now the statement is:

THEOREM 2.1. Let the matrix X be canonical. Then $\lambda_n(X,x)$ equioscillates, i.e., (2.3) $\mu_{2n}(X) = \mu_{3n}(X) = \ldots = \mu_{nn}(X)$,

iff $\Lambda_n(X) = \Lambda_n^*$. Moreover, for arbitrary canonical X

(2.4) $\min_{2\le k\le n} \mu_{kn}(X) \le \Lambda_n^* \le \max_{2\le k\le n} \mu_{kn}(X)$, $n\ge 3$.

Here, the above (so called) optimal matrix X* (which has (2.3)) is unique.

2.3. Using the analogous result of [8] it turns out that the trigonometric interpolation on $[0,2\pi)$ at equidistant nodes is optimal.

For the corresponding Lebesgue constants $\tilde{\Lambda}_n^*$, the values

(2.5) $\tilde{\Lambda}_{2n}^* = \frac{1}{n} \sum_{k=1}^{n} \operatorname{ctg} \frac{2k-1}{4n} \pi$, $n=1,2,\ldots$,

(2.6) $\tilde{\Lambda}_{2n+1}^* = \tilde{\Lambda}_{2(2n+1)}^*$, $n=1,2,\ldots$,

were obtained by H. EHLICH and K. ZELLER [12], (2.4).

The complex case, when the nodes are on the unit circle line Γ, was treated by L. BRUTMAN [10] and L. BRUTMAN, A. PINKUS [11]. They proved that again the case of the equidistant nodes (on Γ) is optimal and the corresponding Lebesgue constants $\bar{\Lambda}_n^*$ are:

(2.7) $\bar{\Lambda}_n^* = \tilde{\Lambda}_{2n}^*$, $n=1,2,\ldots$

(see [10], (23)).

Very recently P.N. SHIVAKUMAR and R. WONG [13] further V.K. DZJADIK and V.V. IVANOV [14] obtained asymptomic expansions for the right side of (2.5) (see further V.K. DZJADIK, S.Ju. DZJADIK and A.S. PRYPIK [19]). Especially, in [13] the expansion

$$(2.8)\quad \tilde{\Lambda}^*_{2n} \overset{A}{\sim} \frac{2}{\pi}\ln n+\chi+\frac{2}{\pi}\ln 2+\sum_{k=1}^{\infty}\frac{a_k}{n^{2k}}\ , \quad a_k = \frac{(-1)^{k+1}(2^{2k-1}-1)^2\pi^{2k-1}B^2_{2k}}{4^{k-1}\ k(2k)!}$$

was established as $n\to\infty$ (B_k are the Bernoulli numbers). Further, the error has the same sign as, and is in absolute value less than, the first term neglected (compare R. GÜNTTNER [18], Theorem 1. and (3.2)). I.e., by (2.5) - (2.8) we see that the problem of the optimal nodes and the corresponding Lebesgue constants is settled considering the trigonometric or the complex interpolation.

3. Asymptotic for Λ^*_n

3.1. If $X\subset[-1,1]$, neither the optimal system, nor Λ^*_n has been known. But there are some estimates for Λ^*_n.
The mentioned Erdős-theorem (see (2.1)) gives a fairly sharp evaluation, especially if we take into account that he could not use Theorem 2.1. and its very useful relation (2.4) (see further (3.1)).

It is easy to see for arbitrary (maybe not canonical) X the next relation

$$(3.1)\quad \min_{1\le k\le n+1}\mu_{kn}(X) \le \min_{2\le k\le n}\mu_{kn}(X)\le\Lambda^*_n\le \max_{2\le k\le n}\mu_{kn}(X)\le \max_{1\le k\le n+1}\mu_{kn}(X)$$

which can be used to obtain estimates for Λ^*_n applying special matrices X and evaluating the differences

$$\delta_n(X) := \max_{2\le k\le n}\mu_{kn}(X) - \min_{2\le k\le n}\mu_{kn}(X)\ ,$$

$$\Delta_n(X) := \max_{1\le k\le n+1} \mu_{kn}(X) - \min_{1\le k\le n+1} \mu_{kn}(X) .$$

3.2. Two very natural choices for the special X are the Chebyshev matrix $T = \{\cos \frac{2k-1}{2n} \pi\}$, $k=1,2,\ldots,n$, $n=1,2,\ldots$ and the matrix of the Chebyshev extremum nodes $V = \{\cos \frac{k\pi}{n-1}\}$, $k=0,1,\ldots,n-1$, $n=2,3,\ldots$.

EHLICH and ZELLER [12] proved that

$$\Lambda_n(T) = \Lambda_{n+1}(V) = \tilde{\Lambda}^*_{2n} \quad , \quad n=1,3,5,\ldots,$$

$$\Lambda_n(T) = \Lambda_{n+1}(V) + \varrho_n = \tilde{\Lambda}^*_{2n} \quad , \quad n=2,4,6,\ldots \; , \; 0<\varrho_n<\frac{1}{n^2} \; ,$$

from where by (2.8)

$$(3.2) \quad \Lambda_n(T) = \frac{2}{\pi} \ln n + \chi + \frac{2}{\pi} \ln 2 + \frac{\pi}{72n^2} - \varepsilon_n$$

where $0<\varepsilon_n<0.0088\, n^{-4}$ if n is big enough. Using analogous estimations and (3.1), DZJADIK and IVANOV [14] got the value of Λ^*_n within the error 0.45.

They had no knowledge on the paper of L. BRUTMAN [15] written in 1978, where using a quite serious analysis of $\lambda_n(T,x)$, he proved that

$$\delta_n(T) < 0.201 \quad \text{if} \quad n \ge 3$$

from where by (3.1) we can obtain Λ^*_n within the error 0.201.

By further analysis R. GÜNTTER [18] obtained that

$$\delta_n(T) = \frac{2}{\pi} \ln 2 - \frac{4}{3\pi} + \sigma(1) \quad ,$$

i.e., the error can be lessen to 0.01686... . But we can not obtain a better estimation for Λ_n^* using T.

Further calculations show that for other special matrices X , $\delta_n(X) > \delta_n(T)$ (see e.g. [9] and [12]).

3.3. Now we are able to prove

THEOREM 3.1. *We have the relations*

$$(3.3) \quad \frac{\text{const}}{(\ln n)^{1/3}} > \Lambda_n^* - \frac{2}{\pi} \ln n - \chi . > \begin{cases} \frac{\pi}{18n^2} + 0\left(\frac{1}{n^4}\right) & \text{if } n=2m, \\ -\frac{1}{2\pi n} + 0\left(\frac{1}{n^2}\right) & \text{if } n=2m+1. \end{cases}$$

3.4. *Some words on the proof*. First, using some properties of the matrix T, we prove the right hand side of (3.3). To obtain the other part is more tedious. Here the main idea is to construct a matrix D which is "close" to T, moreover for which $\Lambda_n(D_c) \approx \frac{2}{\pi} \ln n + \chi$ (D_c is a proper canonical matrix obtained from D.). The surprising fact is that D can be constructed shifting only two proper nodes of T.

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