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DIVERGENCE OF INTERPOLATING QUINTIC SPLINES

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1. Introduction. A function $S(x)$ is called the periodic interpolating spline of degree $2z+1$ on the mesh $\Delta: a=x_0<x_1<\dots<x_N=b$ for the function $f(x)$, if $S(x)$ is $(b-a)$-periodic function, agrees with $f(x)$ at $x_0,x_1,\dots,x_N$, is some polynomial of degree $2z+1$ on each interval of the mesh and $S\in C^{2z}$. Here and below $f(x)$ is $(b-a)$-periodic function.

Let $\{\Delta_\nu: a=x_{\nu 0}<x_{\nu 1}<\dots<x_{\nu N_\nu}=b;\ \nu=1,2,\dots\}$ be a sequence of the meshes with the condition $\bar{h}_\nu=\max_i h_{\nu i}\to 0$, $h_{\nu i}=x_{\nu i+1}-x_{\nu i}$ as $\nu\to\infty$ and denote by ρ_ν the local mesh ratio $\max\{h_{\nu i}/h_{\nu j}: |i-j|=1\}$ The problem of the convergence and divergence of the derivative $S^{(\kappa)}$ of the interpolating spline to the κ-th derivative of $f(x)$ for any periodic $f\in C^\kappa$ on the given sequence of meshes $\{\Delta_\nu\}$ has a great practical importance.

The simplest case, $z=0$, of piecewise linear interpolation is, of course, trivial. In the next simplest case, $z=1$, of cubic spline interpolation, the conditions of the convergence and divergence of interpolation process are known [1] - [3].

We study this problem for quintic spline ($z=2$). We will establish that $S^{(\kappa)}(x)$ can diverge if $\kappa=0,1,4,5$. The results give some information about the quantity of the condition number of matrices, arising from practical construction of the interpolating splines.

2. Divergence. Let the operator $P_\nu^{(\kappa)}: C[a,b]\to L_\infty[a,b]: f^{(\kappa)}\mapsto S_\nu^{(\kappa)}$ transfer the κ-th derivative of $f\in C^\kappa$ in the κ-th derivative of the quintic spline S_ν interpolating $f(x)$ on Δ_ν. We set

$$\|f\|_\infty=\operatorname*{ess\,sup}_{a\le x\le b}|f(x)|\,,\quad \|P_\nu^{(\kappa)}\|=\sup_{\|f^{(\kappa)}\|_\infty=1}\|S_\nu^{(\kappa)}\|_\infty .$$

In accordance with Banach-Stainhause theorem in order to show the divergence of the interpolating splines, it suffices to present an example of the mesh sequence $\{\Delta_\nu\}$, for which sequence $\{\|P_\nu^{(\kappa)}\|\}$ is unbounded.

We construct the sequence $\{\Delta_\nu\}$ similar to [4]. Let q be a fixed positive number, $h_\nu = H_\nu/2(1+q+\dots+q^{\nu-1})$, $H_\nu = (b-a)/\nu$. The knots of Δ_ν we collect as follows

$$a+mH_\nu = x_{\nu,2m\nu} < x_{\nu,2m\nu+1} < \dots < x_{\nu,2m\nu+2\nu} = a+(m+1)H_\nu, \quad m=0,\dots,\nu-1;$$

$$h_{\nu,2m\nu+i} = h_{\nu,2m\nu+2\nu-i-1} = q^i h_\nu, \quad i=0,\dots,\nu-1.$$

It is clear that this mesh sequence satisfies the conditions

$$\lim_{\nu\to\infty} \bar{h}_\nu = 0 \quad , \quad \rho_\nu = \rho \tag{1}$$

with $\rho = \max\{q, q^{-1}\}$.

Let $F_{\nu,i}(x)$ be the fundamental periodic splines on the mesh Δ_ν such that $F_{\nu,i}(x_{\nu j}) = \delta_{ij}$.

Lemma. For $\kappa = 0,\dots,5$ it holds that

$$\|P_\nu^{(\kappa)}\| > K h_\nu^\kappa \Big\| \sum_{m=1}^{\nu} F_{\nu,2m\nu}^{(\kappa)} \Big\|_\infty, \tag{2}$$

with some constant K depending only on κ.

From the uniqueness of the interpolating quintic spline it follows that the function $\sum_{m=1}^{\nu} F_{\nu,2m\nu}(x)$ on $[a, a+H_\nu/2]$ coincides with the quintic spline $\sigma_\nu(x)$ which is defined by the conditions

$$\sigma_\nu(a) = 1, \quad \sigma_\nu(x_{\nu,i}) = 0, \quad i=1,\dots,\nu; \tag{3}$$

$$\sigma_\nu^{(m)}(x_{\nu,p}) = 0, \quad m=1,3; \quad p=0,\nu, \tag{4}$$

on the mesh of the interval $[a, a+H_\nu/2]$.

Let us examine the behaviour $\|\sigma_\nu^{(\kappa)}\|_\infty$ as $\nu\to\infty$. From (2), (3) one finds coefficients α_p of the decomposition of σ_ν over B-spline basis

$$\alpha_p = \omega_2^p\left[C_1\left(\frac{\omega_1}{\omega_2}\right)^p + C_2 + C_3\left(\frac{\omega_3}{\omega_2}\right)^p O\left(\left|\frac{\omega_2}{\omega_3}\right|^\nu\right) + C_4\left(\frac{\omega_4}{\omega_2}\right)^p O\left(\left|\frac{\omega_2}{\omega_4}\right|^\nu\right) + o(1)\right],$$

$$p = -4,\dots,\nu-1,$$

where C_1, C_2, C_4, C_3 are some constants depending on q and $C_2 \neq 0$; $\omega_1, \omega_2, \omega_3, \omega_4$ are the simple, real and negative roots of the equation

$$\frac{1}{q^5}\omega^4+\frac{1}{q}\left(1+\frac{1}{q}\right)\left(4+\frac{5}{q}+\frac{4}{q^2}\right)\omega^3+\left(\frac{6}{q^2}+\frac{16}{q}+22+16q+6q^2\right)\omega^2+$$
$$+q(1+q)(4+5q+4q^2)\omega+q^5=0, \tag{5}$$

with respect to ω.

Consequently, as $\nu\to\infty$

$$\|\sigma_\nu^{(\kappa)}\|_\infty > C\frac{1}{h_\nu^\kappa}\left|\frac{\omega_2}{q^\kappa}\right|^{\nu/2}, \tag{6}$$

with the constant $C>0$ not depending on ν. Now from (2) and (6) one has

$$\|P_\nu^{(\kappa)}\| > \widetilde{K}\left|\frac{\omega_2}{q^\kappa}\right|^{\nu/2}.$$

Equation (5) is equivalent [5] to the equation

$$\sum_{m=0}^{5}\binom{5}{m}(-1)^m\frac{1}{q^m-\omega}=0,$$

for $q\neq 1$. The root ω_2 we can calculate explicitly:

$$\omega_2=-\varphi(q)+\sqrt{\varphi^2(q)-q^2},$$

where $\varphi(q)=q(1+q)\left[q^2+5q/4+1-\sqrt{q^4+q^3+41q^2/16+q+1}\right]$.

Investigating $\omega_2=\omega_2(q)$ as a function on q, we conclude that the inequality $|\omega_2(q)\,q^{-\kappa}|>1$ has the solution for $\kappa=0,1,4,5$. Let q_κ^* be the unique root of the equation $\omega_2(q)=q^\kappa$, $\kappa=0,1,4,5$; $\rho_\kappa^*=\max\{q_\kappa^*,(q_\kappa^*)^{-1}\}$. Here $\rho_0^*=\rho_5^*\approx 1.4$; $\rho_1^*=\rho_4^*\approx 1.8$. Hence

__Theorem 1__. For each fixed $\rho\geq\rho_\kappa^*$, $\kappa=0,1,4,5$, there exist a $(b-a)$-periodic function $f\in C^\kappa$ and a sequence $\{\Delta_\nu\}$ with (1) such that for the periodic quintic splines S_ν, which interpolate in the breakpoints of Δ_ν, the sequence $\{\|S_\nu^{(\kappa)}-f^{(\kappa)}\|_\infty\}$ is unbounded.

Moreover, we know some bounds of ρ_ν, guaranteeing the convergence of splines in the space C and of highest derivatives of splines in the space C^5 (We denote the root of the equation $\omega_2(q)=-q^{-1}$ by $\tilde\rho\approx 1.277$):

__Theorem 2 [6]__. If the sequence $\{\Delta_\nu\}$ satisfies (1) with $\rho<\tilde\rho$ and the quintic splines S_ν interpolate f on Δ_ν, then

i) $\|S_\nu-f\|_\infty\to 0$, all $f\in C$,

ii) $\| S_\nu^{(5)} - f^{(5)} \|_\infty \to 0$, all $f \in C^5$.

3. The condition number of matrices. Let us consider now the problem of practical construction of interpolating periodic quintic spline. The matrices of the equation system, which is to be solved here (the system with respect to knot values of any derivative of the spline or with respect to coefficients of the B-spline decomposition), are not always well conditioned. The well conditioning guarantees the high precision of solution of the system.

Let $f_i = f(x_i)$ be the values of the $(b-a)$-periodic function f in the knots of the mesh $\Delta: a = x_0 < x_1 < \ldots < x_N = b$ of the interval $[a,b]$. Denote

$$h_i = x_{i+1} - x_i \,, \; \rho_\Delta = \max_{|i-j|=1} h_i/h_j \,, \quad \bar{h} = \max_i h_i \,,$$
$$\omega(g;h) = \max \{ |g(x') - g(x'')| : |x' - x''| \le h \,;\, x', x'' \in [a,b] \} .$$

We construct the quintic spline S interpolating f , for example, by means of the B-spline decomposition. It holds the bound

$$\| S - f \|_\infty \le (3 + 2 \|A^{-1}\|)\, \omega(f; \bar{h}) , \tag{7}$$

where $\|A\| = \max_i \sum_j |a_{ij}|$ is the matrix norm, A is the matrix of the knot values of the quintic B-splines on the mesh Δ.

According to theorem 1 there exist a continuous function f and a mesh sequence $\{\Delta_\nu\}$ with the local mesh ratio $\rho_{\Delta_\nu} \ge \rho_0^* \approx 1.4$ such that $\|S - f\|_\infty \to \infty$ and $\bar{h} \to 0$ as $\nu \to \infty$. But in this case $\omega(f; \bar{h}) \to 0$ and according to (7) the sequence of the norm $\|A^{-1}\|$ has to increase unboundedly. Since the condition number of the matrix A is $\mathrm{cond}\,(A) = \|A\| \cdot \|A^{-1}\| = \|A^{-1}\|$, we have

Theorem 3. If the local mesh ratio $\rho_\Delta \ge \rho_0^*$, then the quantity of the condition number of the matrix with respect to the coefficients of the B-spline decomposition can be arbitrarily large.

Similar to this, the matrix of the equation system with respect to the knot values of the κ-th derivative of the spline, $\kappa = 1,4,5$ can also have arbitrarily large the condition number. If the local mesh ratio ρ_Δ satisfies the inequality $\rho_\Delta \ge \rho_\kappa^*$, then the use of the system with respect to the knot values of the spline can lead to considerable loss of the precision of the interpolation.

We guarantee the well conditioning of the matrix of the B-spline system only for $\rho_\Delta < \tilde{\rho} \approx 1.277$ (cf. theorem 2). In order to avoid solving of the systems with bad conditioning matrix in the time of construction of the interpolating quintic spline on an arbitrary nonuniform mesh, we recommend to make use of the systems with respect to the knot values of either second or third derivative.

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