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SOME REMARKS ON APPROXIMABLE ELEMENTS IN GENERALIZED SAKS SPACES

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1. Let (ϱ_i) be a sequence of modulars or pseudomodulars on a linear space X. The spaces defoned by the formulas $X_{\varrho_i}=\{x\in X:\varrho_i(\lambda x)\to 0 \text{ as } \lambda\to 0+\}$ are called the modular spaces or pseudomodular spaces, respectively. Let us suppose that the condition $\varrho_i(x)=0$ for all i implies x=0. By means of the sequence (ϱ_i) we may introduce the following modulars:

$$\varrho_0(x)=\sup_i \varrho_i(x), \qquad \varrho_\sigma(x)=\sup_n \frac{1}{n}\sum_{i=1}^{\infty}\varrho_i(x),$$
(1)
$$\varrho_s(x)=\sum_{i=1}^{\infty}\varrho_i(x), \qquad \varrho_w(x)=\sum_{i=1}^{\infty}2^{-i}\cdot\varrho_i(x)\cdot\left(1+\varrho_i(x)\right)^{-1}.$$

It is well known that the following inclusions are true:

(2) $$X_{\varrho_s}\subset X_{\varrho_0}\subset X_{\varrho_\sigma}\subset X_{\varrho_w}.$$

Definition. Elements $x\in X$ belonging to X_{ϱ_w} (or to $X_{\varrho_\sigma}, X_{\varrho_0}, X_{\varrho_s}$) will be called W-approximable (or σ-approximable, o-approximable or s-approximable) by ϱ_i. In the following we will write briefly: w-appr, σ-appr, o-appr and s-appr, respectively.

Let us remark that by inclusions (2) we have

Remark 1.

(a) Every s-appr element is o-appr and is also σ-appr and w-appr.

(b) Every o-appr element is σ-appr and w-appr.

(c) Every σ-appr element is w-appr.

It is obvious that the converse implications do not hold in general case. In the following we will consider special cases of sequences (ϱ_i) and investigate under which conditions on ϱ_i the opposite implications are true.

2. Let T and T_b denote spaces of all real sequences, bounded real sequences, respectively. Sequences belonging to T will be denoted by $x=(t_\nu)$, etc. In this section $(a_{n\nu})=A$ denotes a non-negative matrix which does not possess column consisting of zeros only and such that $a_{n\nu}\to 0$ as $n\to\infty$ for $\nu=1,2,\dots$ and $a_{n1}+a_{n2}+\dots\leqslant K$ for $n=1,2,\dots$ where $K>0$ is a constant.

Now we define the sequence (ϱ_i) by the formula

$$(3)\qquad \varrho_i(x)=\sup_n\sum_{\nu=1} a_{n\nu}\varphi_i(|t_\nu|)\ ,\quad \text{for } i=1,2,\dots$$

where (φ_i) is a sequence of convex φ-functions. In the following we may introduce new modulars by the formulas (1).

<u>Theorem 1</u>. Every bounded sequence is s-appr if and only if, there holds the following condition:

(1^o) there exists a number u_o such that $\varphi_1(u_o)+\varphi_2(u_o)+\dots<\infty$.

<u>Proof</u>. Let $x\in T_b$ and $L=\sup_\nu|t_\nu|$ and let condition (1^o) holds. Then $\varrho_s(\lambda x)\leqslant\sum_i\varphi_i(\lambda L)\sup_n\sum_\nu a_{n\nu}\to 0$ as $\lambda\to 0+$, becaus $\sum_i\varphi_i(u_o)<\infty$ and $\varphi_i(\lambda L)\leqslant\varphi_i(u_o)$ for $0<L\lambda<u_o$. Thus x is s-appr. Naw, let us suppose that every bounded sequence is s-appr. If we take $x=(1)$, then $\varrho_s(\lambda x)\to 0$ as $\lambda\to 0+$, and so $\varphi_1(\lambda)+\varphi_2(\lambda)+\dots<\infty$ for sufficiently small $\lambda>0$.

<u>Theorem 2.</u> Let us suppose that

(2^o) there existe a $u_o>0$ such that $\varphi_1(u_o)+\varphi_2(u_o)+\dots=\infty$,

every s-appr element is bounded. Let the matrix $(a_{n\nu})$ satisfy the following condition:

(3^o) there exists a sequence of pairwise disjoint sets of indices (B_k) such that $\sup_n\sum_{\nu_j\in B_k}a_{n\nu_j}\to 0$ as $k\to\infty$ and $\sum_{\nu_j\in B_k}a_{n\nu_j}>0$ for all k,

If every s-appr element is bounded then φ-functions φ_i satisfy (2^o).

<u>Proof</u>. If $A_k=\{\nu: k\leqslant|t_\nu|<k+1\}$ then we have

$$\infty>\varrho_s(\lambda x)\geqslant\sum_i\sup_n\sum_k\sum_{\nu\in A_k}a_{n\nu}\varphi_i(\lambda k)\geqslant\sup_n\sum_{\nu\in A_k}a_{n\nu}\sum_i\varphi_i(\lambda k)\ .$$

By the assumption, there exists an index k_o such that $\varphi_1(\lambda k)+\varphi_2(\lambda k)=\infty$ for $k\geqslant k_o$. But $\varrho_s(\lambda x)<\infty$ for some $\lambda>0$, then we have $\sup_n\sum_{\nu\in A_k}a_{n\nu}=0$ for $k\geqslant k_o$. Thus $x\in T_b$. Now, let $a_k=\varphi_1(k)+\varphi_2(k)+\dots$ and let $\varphi_1(u)+\varphi_2(u)+\dots<\infty$ for every $u>0$. We take a subsequence (B_{k_j}) of the sequence (B_k) such that $\sup_n\sum_{\nu\in B_{k_j}}a_{n\nu}<(2^{-j}\cdot a_j)^{-1}$ and we define sequence $x=(t_\nu)$ where $t_\nu=j$ for $\nu\in B_{k_j}$ and $t_\nu=0$ elsewhere. Then $x\notin T_b$ and $\varrho_s(x)\leqslant\sum_j 2^{-j}<\infty$. Hence $\lim_{\lambda\to 0+}\varrho_s(\lambda x)=\sum_i\lim_{\lambda\to 0+}\varrho_i(\lambda x)=0$, i.e. the sequence x is s-appr, a contradiction.

<u>Theorem 3</u>. Every bounded sequence is o-appr if and only if

(4^o) $\varphi_i(u)$ are equicontinuous at $u=0$.

<u>Theorem 4.</u> Let us suppose that

(5^o) there exists a $\bar{u}>0$ such that $\sup\varphi_i(\bar{u})=\infty$, then every o-appr sequence is bounded.

If the conditions (3^o) and (4^o) are satisfy, **then the condition every o-appr element is bounded, implies the condition** (5^o).

Theorem 5. If φ_i satisfy the conditions (2^o) and (4^o) and every o-appr element is also s-appr, then φ_i satisfy the condition (5^o).

Theorem 6. If φ_i satisfy conditions (1^o) and (5^o), **then** every o-appr element is also s-appr.

Theorem 7. Let us suppose that there hold the conditions (4^o),

(6^o) **there exist** positive constants k, c, u_o and i_o such that $\varphi_i(cu) \leqslant k\varphi_{i_o}(u)$ for $0 \leqslant u \leqslant u_o$ and $i \geqslant i_o$,

(7^o) $\liminf_{\nu\to\infty} a_{n_o\nu} > 0$ for a fixed n_o,

then every w-appr element is o-appr.

Theorem 8. Let us suppose that there hold the conditions (4^o),

(8^o) there exists a sequence (ν_j) such that $\lim_{j\to\infty} a_{n\nu_j} > 0$ for a fixed index n_o,

(9^o) for every index i there exist positive constants $\lambda_i, \beta_i, \vartheta_i$ such that for every $u \leqslant \vartheta_i$ and $k \geqslant i$ there holds the inequality $\varphi_i(\lambda_i u) \leqslant \beta_i \varphi_k(u)$,

(10^o) for every $\varepsilon > 0$ there exist numbers $u_\varepsilon > 0$ and $\alpha_\varepsilon > 0$ depending on i such that $\varphi_i(\lambda u) < \varphi_i(u)$ for $0 \leqslant \alpha \leqslant \alpha_\varepsilon$, $u \geqslant u_\varepsilon$.

Every w-appr element is o-appr if and only if, φ_i satisfy the condition (6^o).

3. In this section we take as X the space of real-valued infinitely differentiable functions x(t) on a real line. We define the sequence (ϱ_i) by the formula:

$$(4) \qquad \varrho_i(x) = \int_{-\infty}^{\infty} \varphi(t, |D^{i-1}x(t)|)\,dt \quad , \text{ for } i=1,2,\dots$$

where $D^i x(t)$ is the i-th derivative of x(t) and $\varphi(t,u)$ is a convex φ-function with a parameter t, such that $u^{-1}\cdot\varphi(t,u) \to 0$ as $u \to 0+$ and $u^{-1}\cdot\varphi(t,u) \to \infty$ as $u \to \infty$. In the following by the formulas (1) we introduce new modulars.

Remark 2. **Every** w-appr element is o-appr, if and only if,

$$\sup_i \int_{-\infty}^{\infty} \varphi(t, \lambda_o |D^{i-1}x(t)|)\,dt < \infty \text{ for a } \lambda_o > 0 .$$

Theorem 9. The set of all w-appr elements and the set of all o-appr elements are different for every φ.

Proof. For proof let us remark that if $x \in X_{\varrho_o}$, then either $x \equiv 0$ or the support of x is the dense set in $(-\infty, \infty)$. Thus, there are always w-appr elements which are not o-appr. (for instance $x \in X$,

$x \neq 0$ of compact support).

Let E^1 be the class of entire functions $x(z)$ of a complex variable for which $|x(z)| = O(e^{(1+\varepsilon)|z|})$ for all $\varepsilon > 0$. By E_R^1 we shall denote the class of real-valued functions $x(t)$ on the real line such that each of them can be extended to an entire function of the class E^1 on the complex plane.

Theorem 10. Every σ-appr element belongs to the space E_R^1.

Proof. By the inequality given in [3] we get

$$\varphi(s, |D^i x(s)|) \leq \int_{-\infty}^{\infty} \varphi(t, 2|D^i x(t)|)\,dt + \int_{-\infty}^{\infty} \varphi(t, |D^{i+1} x(t)|)\,dt, \text{ for } i=1,2\ldots$$

Hence $\varphi(s, \frac{\lambda}{2}|D^i x(s)|) \leq \varrho_{i+1}(\lambda x) + \int_{-\infty}^{\infty} \varphi(t, \lambda |D^{i+1} x(t)|)\,dt = \varrho_{i+1}(\lambda x) + \varrho_{i+2}(\lambda x)$ for all $\lambda > 0$, $|s| < \infty$ and all i. Now, by Jensen inequality we have

$$\varphi\Big(t, \frac{\lambda}{2n}\sum_{i=0}^{n-1} |D^i x(t)|\Big) \leq \frac{1}{n}\sum_{i=0}^{n} \varphi(t, \tfrac{\lambda}{2}|D^i x(t)|) \leq 4\,\varrho_\sigma(\lambda x).$$

If $x \in X_{\varrho_\sigma}$, then there exists a number $\lambda > 0$ such that $\varrho_\sigma(\lambda x) < \frac{k}{4} < \infty$ for some $k > 0$. Applying the Lemma from [10] we conclude that there exists a $k > 0$ depending on $K > 0$ such that if $\varphi(t,u) \leq k$ for $t^2 \leq a^2$, then $u \leq K$. In consequence we have $|D^0 x(t)| + |D^1 x(t)| + \ldots + |D^{n-1} x(t)| \leq \leq nK$ for $|t| < a < \infty$ and in particular $|D^i x(t)| \leq (i+1)K$ for all i and $|t| < a < \infty$.

Now, by the same way as in [2] we have condition $|x(z)| = O(e^{(1+\varepsilon)|z|})$ for every $\varepsilon > 0$ and for sufficiently large $|z|$.

Remark 3. $X_{\varrho_\sigma} \subset E_R^1 \cap X_{\varrho_w}$ and $X_{\varrho_0} \subset E_R^1 \cap X_{\varrho_w}$.

Theorem 11. Every o-appr element belongs to the space E_R^1.

Proof. If $x \in X_{\varrho_0} = \mathcal{D}_\varphi^0$ then applying Theorem from [12] we have that all its derivatives are uniformly bounded. In consequence x may be developed in the Taylor series and extended to the whole complex plane.

4. Let X be a real or complex linear space and let two sequences of convex pseudomodulars or modulars (ϱ_i) and (ϱ_i') be given in X. Let us suppose that $\varrho_i(x)=0$ for all i implies $x=0$ and $\varrho_i'(x) = 0$ for all i implies $x=0$.

By means of these sequences and by formulas (1) one may define the following modulars

(5) $\varrho_0, \varrho_s, \varrho_\sigma, \varrho_w$ and $\varrho_0', \varrho_s', \varrho_\sigma', \varrho_w'$.

In the following $\bar{\varrho}$ and $\bar{\varrho}'$ denote one of these modulars $\varrho_0, \varrho_s, \varrho_\sigma, \varrho_w$ or $\varrho_0, \varrho_s, \varrho_\sigma, \varrho_w$, respectively.

Now we shall consider the generalized Saks spaces

(6) $$X_S(\bar{\varrho},\bar{\varrho}') = \langle \{x \in X : \bar{\varrho}(x) \leqslant 1\},\ \bar{\varrho}' \rangle$$

where $\bar{\varrho}$ and $\bar{\varrho}'$ are defined as above and X is a given space.

Convergence of the sequence (x_n) of elements of generalized Saks space $X_S(\bar{\varrho},\bar{\varrho}')$ to the element x of the space $X_S(\bar{\varrho},\bar{\varrho}')$ will mean that $x_n \in X_S(\bar{\varrho},\bar{\varrho}')$ and $x_n \xrightarrow{\bar{\varrho}'} x$.

It is easy to verify that for an arbitrary sequence (x_n), $x_n \in X$ the following conditions hold:

Remark 4. ϱ'_s-convergence implies ϱ'_o-convergence,
ϱ'_o-convergence implies ϱ'_σ-convergence,
ϱ'_o-convergence implies ϱ'_w-convergence.

Remark 5. ϱ_s-boundedness implies ϱ_o-boundedness,
ϱ_o-boundedness implies ϱ_σ-boundedness,
ϱ_o-boundedness implies ϱ_w-boundedness.

Let the generalized Saks spaces $X_S(\bar{\varrho}_i,\bar{\varrho}'_i)$, i=1,2 be given. If $\{x \in X : \bar{\varrho}_1(x) \leqslant 1\} \subset \{x \in X : \bar{\varrho}_2(x) \leqslant 1\}$ and convergence of the sequence (x_n) to x in $X_s(\bar{\varrho}_1,\bar{\varrho}'_1)$ implies its convergence to x in the space $X_S(\bar{\varrho}_2,\bar{\varrho}'_2)$, then we write it as

(7) $$X_S(\bar{\varrho}_1,\bar{\varrho}'_1) \hookrightarrow X_S(\bar{\varrho}_2,\bar{\varrho}'_2) .$$

For two sequences of pseudomodulars or modulars (ϱ_i) and (ϱ'_i) we have the following examples of theorems:

Theorem 12. $X_S(\bar{\varrho},\varrho'_s) \hookrightarrow X_S(\bar{\varrho},\varrho'_o) \hookrightarrow X_S(\bar{\varrho},\varrho'_w)$,
$X_S(\bar{\varrho},\varrho'_o) \hookrightarrow X_S(\bar{\varrho},\varrho'_\sigma)$,
$X_S(\varrho_s,\bar{\varrho}') \hookrightarrow X_S(\varrho_o,\bar{\varrho}') \hookrightarrow X_S(\varrho_\sigma,\bar{\varrho}')$,
$X_S(\varrho_o,\bar{\varrho}') \hookrightarrow X_S(\varrho_w,\bar{\varrho}')$.

Theorem 13. $X_S(\varrho_o,\varrho'_o) \hookrightarrow X_S(\varrho_w,\varrho'_w)$,
$X_S(\varrho_o,\varrho'_o) \hookrightarrow X_S(\varrho_\sigma,\varrho'_\sigma)$,
$X_S(\varrho_s,\varrho'_s) \hookrightarrow X_S(\varrho_o,\varrho'_s) \hookrightarrow X_S(\varrho_\sigma,\varrho'_o) \hookrightarrow X_S(\varrho_\sigma,\varrho'_w)$.

Now we shall give examples of opposite relations. It is possible only for special sequences (ϱ_i) and (ϱ'_i) , where

(8) $$\varrho_i(x) = \sup_n \sum a_{n\nu}\varphi_i(|t_\nu|), \qquad \varrho'_i(x) = \sup_n \sum a_{n\nu}\psi_i(|t_\nu|)$$

i=1,2,... and where φ_i, ψ_i, T, T_b and A have the same meaning as in part 2.

Theorem 14. Let the φ-functions φ_i and ψ_i satisfy conditions (4°),(6°) and let the matrix $(a_{n\nu})$ possess the property (7°). The following conditions hold:

$$X_S(\bar{\varrho},\varrho'_w) \hookrightarrow X_S(\bar{\varrho},\varrho'_o) ,$$
$$X_S(\varrho_w,\bar{\varrho}') \hookrightarrow X_S(\varrho_o,\bar{\varrho}') .$$

Theorem 15. If φ-functions φ_i and ψ_i satisfy conditions (1°),(5°),(4°) , then we have

$$X_S(\bar{\varrho},\varrho'_o) \hookrightarrow X_S(\bar{\varrho},\varrho'_s) ,$$
$$X_S(\varrho_o,\bar{\varrho}') \hookrightarrow X_S(\varrho_s,\bar{\varrho}') .$$

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