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LIMITS OF ALGEBRAIC SUBSETS OF $\mathbb{C}^n$

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1. Topology of local uniform convergence. Let X be a metric space. Let $\mathcal{F}_X$ be the family of all closed subsets of X. We endow $\mathcal{F}_X$ with the topology $\mathcal{T}_X$ generated by the sets

$$\mathcal{U}(S,K) = \{F \in \mathcal{F}_X \mid F \cap K = \emptyset \text{ and } F \cap U \neq \emptyset \text{ for } U \in S\}$$

corresponding to all compact subsets $K \subset X$ and all finite families S of open subsets of X. We call this topology the topology of local uniform convergence. We will write $F_\nu \longrightarrow F$ if F is the limit set of the sequence $\{F_\nu\}$ in the topology of local uniform convergence.

Now, we recall one simple lemma (for a complete discussion of this topology see [4]).

LEMMA 1. If $F_\nu, F \in \mathcal{F}_X, \nu = 1,2,\ldots$ then the following statements are equivalent

(a) $F_\nu \longrightarrow F$.

(b) Every point $x \in F$ is a limit point of a sequence $\{x_\nu\}$ such that $x_\nu \in F_\nu$ for $\nu = 1,2,\ldots$ and for every $x \in X \setminus F$ there exists a neighborhood U of x such that $F_\nu \cap U = \emptyset$ for almost all indices ν.

Let N_1, N_2 be two finite dimensional ,complex vector spaces. Let Ω be an open subset of N_1. Let $f : \Omega \longrightarrow N_2$ be a mapping. Throught this note we will identify f with its graph. Thus, if f is continuous then f can be regarded as an element of $\mathcal{F}_{\Omega \times N_2}$.

Lemma 2. Let $f_\nu, f: \Omega \longrightarrow N_2$ be continuous mapping for $\nu = 1,2,\ldots$ Then the sequence $\{f_\nu\}$ converges uniformly to f on compact subsets of Ω if and only if $f_\nu \longrightarrow f$ in the topology $\mathcal{T}_{\Omega \times N_2}$.

Proof. If the sequence $\{f_\nu\}$ converges to f uniformly on compact subsets of Ω then by virtue of Lemma 1 the sequence $\{f_\nu\}$ converges to f in the topology $\mathcal{T}_{\Omega \times N_2}$.

Conversly, let us suppose that $f_\nu \longrightarrow f$ in the topology $\mathcal{T}_{\Omega\times N_2}$. With $x_0 \in \Omega$ let us associate an open, connected neighborhood U of x_0 such that $\overline{U}$ is a compact and $\overline{U} \subset \Omega$. For every $\varepsilon > 0$ we define a neighbourhood $\mathcal{U}$ of f by

$$\mathcal{U} = \mathcal{U}(\{(x,y) \in U\times N_2 \mid |f(x)-y| < \varepsilon\}, \{(x,y) \in \overline{U}\times N_2 \mid |f(x)-y| = \varepsilon\}).$$

Since U is connected, we conclude that if $f_\nu \in \mathcal{U}$ then $|f_\nu(x) - f(x)| < \varepsilon$. Thus convergence of $\{f_\nu\}$ in the topology $\mathcal{T}_{\Omega\times N_2}$ implies uniform convergence of $\{f_\nu\}$ on compact subsets of Ω.

Let Ω be an open subset of $\mathbb{C}^N$. By $A_p(\Omega)$ we will denote the subspace of $\mathcal{F}_\Omega$ consisting of all purely p-dimensional analytic subsets of Ω, the empty set includet for p=0,...,n.

THEOREM 1. (see [3], Th.3.) Let $V_0 \in A_p(\Omega)$, $W_0 \in A_q(\Omega)$ and $p+q \geq N$. If $V_0 \cap W_0 \in A_{p+q-N}(\Omega)$ then the mapping

$$\cap : A_p(\Omega) \times A_q(\Omega) \ni (V,W) \longrightarrow V \cap W \in \mathcal{F}_\Omega$$

is continuous at the point (V_0, W_0).

We end this section with a simple corollary of Theorem 1.

COROLLARY. (maximum principle, cp. [5], p.55-56) Let U be an open, connected subset of $\mathbb{C}^n$. Let $f : U \longrightarrow \mathbb{C}^m$ be a holomorphic mapping. If there exist affine hyperplanes H, H_ν, $\nu = 1,2,\ldots$ in $\mathbb{C}^m$ such that $f(U) \cap H \neq \emptyset$, $f(U) \cap H_\nu = \emptyset$ for $\nu = 1,2,\ldots$ and $H_\nu \longrightarrow H$ then $f(U) \subset H$.

Proof. Let a be a point of U such that $f(a) \in H$. If for every affine complex line l throught a we have $f(l) \cap H \subset H$. From this it follows that we may assume U to be a bounded, connected, open subset of $\mathbb{C}$. Then f is a local, irreducible subset of $\mathbb{C} \times \mathbb{C}^m$ of dimension one such that $(a, f(a)) \in f \cap (\mathbb{C} \times H)$. Hence $(a, f(a))$ is an isolated point of $f \cap (\mathbb{C} \times H)$ or $f \cap (\mathbb{C} \times H) = f$. If $(a, f(a))$ is an isolated point of $f \cap (C \times H)$ then in virtue of Theorem 1 there exists a neighbourhood $\mathcal{U}$ of H such that $f \cap (\mathbb{C} \times H') \neq \emptyset$ for $H' \in \mathcal{U}$ and this contradicts our assumption.

<u>2. Limits of algebraic sets.</u> E. Bishop ([1], see also [2]) has proved the following theorem.

THEOREM 2 (Bishop) Let $\{V_\nu\}$ be a sequence of purely k-dimensional analytic subets of an open subset D of $\mathbb{C}^n$ which $\mathcal{T}_D$-converges to a (non-empty) limit set V. If for every compact set $K \subset D$, 2k-volumes of $V_\nu \cap K$ are uniformly bounded then V is again a purely k-dimensional analytic subset of D.

Let $V_{k,d}$ be the subset of $\mathcal{F}_{\mathbb{C}^n}$ consisting of all purely k-dimensional algebraic subsets of $\mathbb{C}^n$ of degree $\leqslant d$. It has proved ([4], Th.2) that $V_{k,d}$ is a compact subset of $\mathcal{F}_{\mathbb{C}^n}$. We see at once that this theorem covers the well-known theorem stating that the limit of a sequence of polynomials of degree $\leqslant d$ is again a polynomial of degree $\leqslant d$.

As it is well-known for every entire function f there exists a sequence of polynomials with the limit f (in topology of local uniform convergence). Therefore the following question arises : does for ewery analytic variety $X \subset \mathbb{C}^n$ there exists a sequence of algebraic sets (or algebraic varieties) convergent to X ?

Now, we shall prove the following

THEOREM 3. Let $F_j : \mathbb{C}^n \longrightarrow \mathbb{C}$ be an entire function for $j = 1,\dots,n-k$, $n > 1$, $1 \leqslant k < n$. If X is a pure k-dimensional subset of $\mathbb{C}^n$ defined by

$$X = \{x \in \mathbb{C}^n \mid F_1(x) = \dots = F_{n-k}(x) = 0\}$$

then there exists a sequence $\{V_\nu\}$ of algebraic subsets of $\mathbb{C}^n$ of pure dimensions k, convergent to X.

Proof. We shall identify $F = (F_1,\dots,F_{n-k} : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-k}$ with the graph of F.

Let us observe that $X \times \{0\} = F \cap (\mathbb{C}^n \times \{0\})$.

Since F is an entire mapping there exists a sequence $\{F_\nu\}$ of polynomial mappings which converges (locally uniformly) to F. Following Theorem 1 on continuity of the intersection we see that the sequence

$$X_\nu = \{x \in \mathbb{C}^n \mid F_\nu(x) = 0\}, \ \nu = 1,2,\dots$$

tend, locally uniformly, to X.

Therefore I only know the answer to this question in case of complete intersections. It is an open problmm whether Theorem 3 is true without the assumption of complete intersection ?

References

1. E.Bishop,Condition for the analyticity of certain sets,Mich.Math. Jour.,11 (1964),289-304.
2. G.Stolzenberg,Volumes Limits and Extension of Analytic Varieties, Lecture Notes in Mathematics,Springer,Berlin-Heidelberg-New York 19 (1966).
3. P.Tworzewski,T.Winiarski,Continuity of intersection of analytic sets Annales Polonici Mathematici,42 (1983),387-393.
4. P.Tworzewski,T.Winiarski,Limits of algebraic sets,24 (1984), 151-153.
5. S.W.Šabat,Wwiedienije w kompleksnyj analiz II,Moskwa 1976.

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