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MODULI OF SMOOTHNESS ASSOCIATED WITH CHEBYSHEV SYSTEMS AND APPROXIMATION BY L-SPLINES

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1. Introduction. The following property of the modulus of smoothness of order n $\omega_n(f,h)$ is known: $\omega_n(P_{n-1},h) = 0$ for any polynomial P_{n-1} of degree at most n-1. Let $U = \{u_i\}_{i=o}^{n-1}$ be an extended complete Chebyshev system (ECT-system) in the interval $I = [0,1]$ and let $L = D^n + \sum_{i=0}^{n-1} a_i(t)D^i$ be a linear differential operator with the null space N_L which is a linear span over the system U. The purpose of this paper is to generalize the modulus of smoothness of order n to a modulus of smoothness w.r.t. the operator L (the system U) $\omega_L(f,h)$ such that $\omega_L(u,h) = 0$ for any $u \in N_L$. It appears that the majority of properties of the moduli $\omega_n(f,h)$ hold for the moduli $\omega_L(f,h)$. Further we shall generalize the H.Whitney theorem [11], the Freud - Popov lemma [3] and then we shall obtain theorems of Jackson type for the approximation by L-splines.

2. Extended complete Chebyshev systems and divided differences (cf.[4,12]).

The system $U = \{u_i\}_{i=o}^{n-1}$, $u_i \in C^n(I)$ is called an ECT-system in I if for any points $0 \leqslant t_o \leqslant t_1 \leqslant \ldots \leqslant t_k \leqslant 1$, $k = 0,\ldots,n-1$

$$D\begin{pmatrix} u_o,\ldots,u_k \\ t_o,\ldots,t_k \end{pmatrix} = \det\left[D^{d_j} u_i(t_j)\right]_{i,j=o}^{k} > 0,$$

where $d_j = \max\{l: t_j = t_{j-1} = \ldots = t_{j-l}\}$, $j = 0,\ldots,k$, and D is the differentiation operator.

An ECT-system U admits the representation

$$u_o(t) = w_o(t)$$

$$(1)\qquad u_i(t) = w_o(t)\int_0^t w_1(\tau_1)\int_0^{\tau_1} w_2(\tau_2)\ldots\int_0^{\tau_{i-1}} w_i(\tau_i)\,d\tau_i\ldots d\tau_1,$$

$i = 1,\ldots,n-1$, where $w_i \in C^{n-i}(I)$, $w_i > 0$ for $t \in I$, $i = 0,\ldots,n-1$.

The adjoint system $V = \{v_i\}_{i=0}^{n-1}$ is defined as follows:

$$v_0(t) = 1$$

(2)

$$v_i(t) = \int_0^t w_{n-1}(\tau_1) \int_0^{\tau_1} w_{n-2}(\tau_2) \ldots \int_0^{\tau_{i-1}} w_{n-i}(\tau_i)\, d\tau_i \ldots d\tau_1,$$

$i = 1,\ldots,n-1$.

Define $D_j f(t) = \frac{d}{dt}\frac{f(t)}{w_j(t)}$, $D_j^* f(t) = \frac{1}{w_j(t)}\frac{d}{dt}f(t)$, $j = 0,\ldots,n-1$,

$Lf = D_{n-1}\ldots D_0 f$ and $L^* f = D_0^* \ldots D_{n-1}^* f$.

The systems (1) and (2) span the null spaces of the differential operators L and L^* respectively.

Let $\Delta = \{0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_N = 1\} = \{0 = s_0 < s_1 < \ldots < s_M = 1\}$,

where $t_0,\ldots,t_N = \overbrace{s_0,\ldots,s_0}^{\alpha_0},\ldots,\overbrace{s_M,\ldots,s_M}^{\alpha_M}$ and α_j is the multiplicity of the point s_j, $j = 0,\ldots,M$, $\sum_{j=0}^{M}\alpha_j = N+1$.

A function s is called an <u>L-spline</u> w.r.t. the partition Δ if

(a) $Ls = 0$ in the intervals (s_{j-1}, s_j), $j = 1,\ldots,M$,

(b) $\exists_{\varepsilon>0}: s \in C^{n-1-\alpha_j}(s_j-\varepsilon, s_j+\varepsilon)$, $j = 1,\ldots,M-1$.

We denote the set of these functions by $S_\Delta^L(I)$.

We define the <u>divided difference</u> of a function f at the points $t_0 \leqslant \ldots \leqslant t_n$, $t_0 < t_n$ w.r.t. the operator L (the system U) by

$$[t_0,\ldots,t_n;f]_L = \frac{D\begin{pmatrix} u_0,\ldots,u_{n-1},f \\ t_0,\ldots,t_{n-1},t_n \end{pmatrix}}{D\begin{pmatrix} u_0,\ldots,u_{n-1},u_n \\ t_0,\ldots,t_{n-1},t_n \end{pmatrix}},$$

where u_n is any function satisfying the equation $Lu = 1$.

We may put $w_n = 1$ and define u_n by (1) (see [7,8,12]). It follows from the definition that the divided difference does not depend on the choice of a basis of the space N_L.

Let M_i be the i^{th} L^*B-spline (basic spline w.r.t. the system V and the partition Δ) i.e. the function satisfying the following conditions: 1^o $M_i \in S_\Delta^{L^*}(I)$, 2^o supp $M_i = [t_i,\ldots,t_{i+n}]$, $\int_0^1 M_i(t)dt = 1$ (see [7,8,12]). Then

(3) $$[t_0,\ldots,t_n;f]_L = \int_{t_0}^{t_n} Lf(t)M_0(t)dt.$$

Applying this equality we may prove the following

Theorem 1. (see [12]) Let $\Delta = \{0 \leqslant t_o \leqslant \dots \leqslant t_n \leqslant 1\}$ and $\Delta' = \{0 \leqslant t'_o < \dots < t'_n \leqslant 1\}$ and let $f \in C^n(I)$. Then there exists

$$\lim_{\substack{t'_i \to t_i \\ i=0,\dots,n}} [t'_o,\dots,t'_n;f]_L = [t_o,\dots,t_n;f]_L .$$

3. Interpolation by generalized polynomials. Let $f \in C(I)$ and $0 \leqslant t_o < t_1 < \dots < t_{n-1} \leqslant 1$. Then there exists a unique polynomial P_{n-1} w.r.t. the system U interpolating the function f at the points t_i, $i = 0,\dots,n-1$. We may write this polynomial as follows:

$$(4)\qquad P_{n-1}(t) = -\frac{D\begin{pmatrix} g,u_o,\dots,u_{n-1} \\ t,t_o,\dots,t_{n-1}\end{pmatrix}}{D\begin{pmatrix} u_o,\dots,u_{n-1} \\ t_o,\dots,t_{n-1}\end{pmatrix}} = \sum_{j=0}^{n-1} f(t_j)W_j(t),$$

where g is any function such that $g(t) = 0$, $g(t_j) = f(t_j)$, $j = 0,\dots,n-1$ and W_j is a polynomial w.r.t. the system U satisfying the following conditions: $W_j(t_i) = \delta_{ij}$, $i,j = 0,\dots,n-1$.
Further

$$f(t) - P_{n-1}(t) = \frac{D\begin{pmatrix} f,u_o,\dots,u_{n-1} \\ t,t_o,\dots,t_{n-1}\end{pmatrix}}{D\begin{pmatrix} u_o,\dots,u_{n-1} \\ t_o,\dots,t_{n-1}\end{pmatrix}} =$$

$$(5)\qquad = \frac{D\begin{pmatrix} f,u_o,\dots,u_{n-1} \\ t,t_o,\dots,t_{n-1}\end{pmatrix}}{D\begin{pmatrix} u_n,u_o,\dots,u_{n-1} \\ t\ ,t_o,\dots,t_{n-1}\end{pmatrix}} \cdot \frac{D\begin{pmatrix} u_n,u_o,\dots,u_{n-1} \\ t\ ,t_o,\dots,t_{n-1}\end{pmatrix}}{D\begin{pmatrix} u_o,\dots,u_{n-1} \\ t_o,\dots,t_{n-1}\end{pmatrix}} = [t,t_o,\dots,t_{n-1};f]_L \cdot W(t)$$

where W is a polynomial w.r.t. the system $\{u_o,\dots,u_n\}$ equal to 0 at the points t_j, $j = 0,\dots,n-1$ such that $LW = 1$ and u_n is any function satisfying the equation $Lu = 1$.
Hence by (3)

$$f(t) - P_{n-1}(t) = W(t)\int_0^1 Lf(x)M(x)\,dx ,$$

where M is the L^* B-spline defined for the points $t,t_o,\dots,t_{n-1}$. Let $\widetilde{W}$ be a polynomial w.r.t. the system $\{t^i\}_{i=0}^n$ such that $D^n\widetilde{W} = 1$ and $\widetilde{W}(t_j) = 0$, $j = 0,\dots,n-1$. Put $\{x_o,\dots,x_n\} = \{t,t_o,\dots,t_{n-1}\}$, $x_o < x_1 < \dots < x_n$. Hence by (5) $W(t) = \frac{L}{M}$ and $L = \det[u_i(x_j)]_{i,j=0}^n$.
We may assume that $u_o = 1$. Further

$$u_i(x_j) = \int_0^{x_j} w_1(\tau_1) \int_0^{\tau_1} w_2(\tau_2) \dots \int_0^{\tau_{i-1}} w_i(\tau_i) d\tau_i \dots d\tau_1 .$$

Substracting the j^{th} column from its successor, afterward expanding about the first row and applying properties of determinants we obtain

$L = \int_{x_0}^{x_1} w_1(y_1) \dots \int_{x_{n-1}}^{x_n} w_1(y_n) \det[a_{ij}]_{i,j=1}^{n-1} dy_1 \dots dy_n$, where $a_{1j} = 1$,

$a_{ij} = \int_0^{y_j} w_2(\tau_2) \int_0^{\tau_2} w_3(\tau_3) \dots \int_0^{\tau_{i-1}} w_i(\tau_i) d\tau_i \dots d\tau_2$, $i = 2,\dots,n$, $j =$ $= 1,\dots,n$. Let $\tilde{L} = \det[t_j^i]_{i,j=0}^n$ and $\tilde{W} = \frac{\tilde{L}}{\tilde{M}}$. For the system $\{t^i\}_{i=0}^n$, $\tilde{w}_0 = 1$, $\tilde{w}_i = i$, $i \geq 1$. Because $w_i \in C(I)$ and $w_i > 0$ then there exist positive constants c_i and d_i such that $c_i \tilde{w}_i \leq w_i \leq d_i \tilde{w}_i$. Applying this inequality we prove by induction that $c' \tilde{L} \leq L \leq d' \tilde{L}$, where $c' = \prod_{j=1}^n c_j^{n+1-j}$, $d' = \prod_{j=1}^n d_j^{n+1-j}$. Estimating M analogously, we obtain

<u>Lemma 1.</u> There exist positive constants c and d depending only on the system U such that

$$c|\tilde{W}(t)| \leq |W(t)| \leq d|\tilde{W}(t)|. \tag{6}$$

Hence we obtain

<u>Theorem 2.</u> Let P_{n-1} be a polynomial w.r.t. the system U interpolating a given function $f \in C^n(I)$ at the points $t_0 < t_1 < \dots < t_{n-1}$. Then

$$|f(t) - P_{n-1}(t)| \leq C_U \|Lf\|_\infty |t - t_0| \cdot \dots \cdot |t - t_{n-1}|, \quad t \in I,$$

where C_U is a constant depending only on the system U.

We may write (4) in the following form:

$$P_{n-1}(t) = a_0 u_0(t) + \sum_{j=1}^{n-1} a_j \frac{D\begin{pmatrix} u_0,\dots,u_{j-1},u_j \\ t_0,\dots,t_{j-1},t \end{pmatrix}}{D\begin{pmatrix} u_0,\dots,u_{j-1} \\ t_0,\dots,t_{j-1} \end{pmatrix}}.$$

Put

$$\begin{bmatrix} u_0,\dots,u_j \\ t_0,\dots,t_j \end{bmatrix} f \Big] = \frac{D\begin{pmatrix} u_0,\dots,u_{j-1},f \\ t_0,\dots,t_{j-1},t_j \end{pmatrix}}{D\begin{pmatrix} u_0,\dots,u_j \\ t_0,\dots,t_j \end{pmatrix}}, \quad j = 1,\dots,n.$$

Hence

$$\left[\begin{matrix} u_0,\dots,u_j \\ t_0,\dots,t_j \end{matrix} \middle| f\right] = \left[\begin{matrix} u_0,\dots,u_j \\ t_0,\dots,t_j \end{matrix} \middle| P_{n-1}\right] = a_j \left[\begin{matrix} u_0,\dots,u_j \\ t_0,\dots,t_j \end{matrix} \middle| u_j\right] = a_j$$

and we obtain

$$(7) \qquad P_{n-1}(t) = \frac{f(t_o)}{u_o(t_o)} u_o(t) + \sum_{j=1}^{n-1} \begin{bmatrix} u_o,\dots,u_j \\ t_o,\dots,t_j \end{bmatrix} f \Bigg] \frac{D\begin{pmatrix} u_o,\dots,u_{j-1},u_j \\ t_o,\dots,t_{j-1},t \end{pmatrix}}{D\begin{pmatrix} u_o,\dots,u_{j-1} \\ t_o,\dots,t_{j-1} \end{pmatrix}} .$$

For the system $\{t^i\}_{i=o}^{n}$ we obtain the Newton interpolation formula and because of this, we shall call the formula (7) the Newton interpolation formula.

We may also calculate the coefficients a_j with the help of the following

Theorem 3. (Mühlbach [5]). Let $\{u_o,\dots,u_n\}$, $\{u_o,\dots,u_{n-1}\}$ and $\{u_o,\dots,u_{n-2}\}$ be Chebyshev systems over I. Consider n+1 different points $t_i \in I$, $i = 0,\dots,n$. Then

$$\left[\begin{matrix} u_o,\dots,u_n \\ t_o,\dots,t_n \end{matrix}\middle| f\right] = \frac{\left[\begin{matrix} u_o,\dots,u_{n-1} \\ t_1,\dots,t_n \end{matrix}\middle| f\right] - \left[\begin{matrix} u_o,\dots,u_{n-1} \\ t_o,\dots,t_{n-1} \end{matrix}\middle| f\right]}{\left[\begin{matrix} u_o,\dots,u_{n-1} \\ t_1,\dots,t_n \end{matrix}\middle| u_n\right] - \left[\begin{matrix} u_o,\dots,u_{n-1} \\ t_o,\dots,t_{n-1} \end{matrix}\middle| u_n\right]} .$$

Let now $t_o < t_1 < \dots < t_{n-1}$ and let l_i be the fundamental Lagrange polynomials of degree n-1 defined for the points t_j i.e. $l_i(t_j) = \delta_{ij}$, $i,j = 0,\dots,n-1$. Analogously as Lemma 1 we can prove

Lemma 2. There exist positive constants α and β depending only on the system U such that

$$(8) \qquad \alpha|l_i(t)| \leqslant |W_i(t)| \leqslant \beta|l_i(t)|, \quad t \in I,$$

where the functions W_i are defined by (4).

Remark. Applying Theorem 1 we may extend the facts given above to partitions with multiple points as well.

4. Moduli of smoothness associated with an ECT-system. Let $f \in C(I)$ and let U and the operator L be defined as in the point 2. Put $\Delta_h^L f(t) = (n-1)!h^n[t,t+h,\dots,t+nh;f]_L$. Let q be a polynomial w.r.t. the system U interpolating the function f at the points t+jh, j = = 1,...,n. Then by (5) and (6) we obtain

$$\alpha'|\Delta_h^L f(t)| \leqslant |f(t) - q(t)| \leqslant \beta'|\Delta_h^L f(t)|,$$

where the constants α' and β' depend only on the system U.

We define the modulus of smoothness of the function f w.r.t. the system U (operator L) by the formula

$$\omega_L(f,\delta) = \sup\{|\Delta_h^L f(t)|,\ 0 < h \leqslant \delta,\ t,\ t+nh \in I\}.$$

If $f \in L_p(I)$ for $1 \leqslant p < \infty$, we put

$$\omega_L^{(p)}(f,\delta) = \sup_{0<h\leq\delta}\left(\int_0^{1-nh} |\Delta_h^L f(t)|^p dt\right)^{1/p}.$$

For the operator $L = D^n$ we obtain the modulus of smoothness of order n.

We shall prove the following properties of the moduli of smoothness:

(P.1) $\quad 0 \leq \omega_L(f,\delta) \leq \omega_L(f,\delta')$ for $\delta \leq \delta'$.

(P.2) $\quad \omega_L(f,\delta) \leq c\|f\|_\infty,$

where the constant c depends only on the system U.

(P.3) $\quad \omega_L(f+g,\delta) \leq \omega_L(f,\delta) + \omega_L(g,\delta)$.

(P.4) $\quad \omega_L(f,m\delta) \leq m^n\omega_L(f,\delta)$.

(P.5) $\quad \omega_L(f,\lambda\delta) \leq (1+\lambda)^n\omega_L(f,\delta)$.

(P.6) $\quad \dfrac{\omega_L(f,\delta_1)}{\delta_1^n} \leq 2^n \dfrac{\omega_L(f,\delta)}{\delta^n}$, for $0<\delta\leq\delta_1$.

(P.7) If $f\in C(I)$ and $\omega_L(f,\delta) = o(\delta^n)$ by $\delta\to 0+$, then f is a polynomial w.r.t. the system U.

(P.8) $\quad \lim_{\delta\to 0} \omega_L(f,\delta) = 0$, for $f\in C(I)$.

To prove these properties, we need the following

Theorem 4. Let $\Delta = \{0\leq t_0 < t_1 < \ldots < t_N \leq 1\}$ be a given partition of I, $t_0 \leq t_{k_0} < t_{k_1} < \ldots . < t_{k_n} \leq t_N$. Then there exist numbers α_j, $0<\alpha_j<1$ such that $\sum_{j=k_0}^{k_n-n}\alpha_j = 1$ and for any function f defined on I

$$[t_{k_0},\ldots,t_{k_n};f]_L = \sum_{j=k_0}^{k_n-n}\alpha_j[t_j,\ldots,t_{j+n};f]_L. \tag{9}$$

Proof. (9) is obvious for n = N. Let us assume that it holds for a partition Δ' obtained from Δ by omission a point x of it. Put $x_j = t_{k_j}$, $x \neq x_j$, $j = 0,\ldots,n$, $x_0 < x < x_n$. Applying Theorem 3 we obtain

$$\begin{bmatrix} u_0,\ldots,u_n \\ x_0,\ldots,x_n \end{bmatrix} f\Bigg] = \frac{\begin{bmatrix} u_0,\ldots,u_{n-1} \\ x_1,\ldots,x_n \end{bmatrix} f\Bigg] - \begin{bmatrix} u_0,\ldots,u_{n-1} \\ x_0,\ldots,x_{n-1} \end{bmatrix} f\Bigg]}{\begin{bmatrix} u_0,\ldots,u_{n-1} \\ x_1,\ldots,x_n \end{bmatrix} u_n\Bigg] - \begin{bmatrix} u_0,\ldots,u_{n-1} \\ x_0,\ldots,x_{n-1} \end{bmatrix} u_n\Bigg]} = \frac{L}{M}.$$

Further

$$L = \left(\begin{bmatrix} u_0,\ldots,u_{n-1} \\ x_1,\ldots,x_n \end{bmatrix} f\Bigg] - \begin{bmatrix} u_0,\ldots,u_{n-2},u_{n-1} \\ x_1,\ldots,x_{n-1},x \end{bmatrix} f\Bigg]\right) +$$

$$+ \left(\begin{bmatrix} u_0,\ldots,u_{n-2},u_{n-1} \\ x_1,\ldots,x_{n-1},x \end{bmatrix} f\Bigg] - \begin{bmatrix} u_0,\ldots,u_{n-1} \\ x_0,\ldots,x_{n-1} \end{bmatrix} f\Bigg]\right) =$$

$$= \left(\begin{bmatrix} u_0,\ldots,u_{n-1} \\ x_1,\ldots,x_n \end{bmatrix} u_n\Bigg] - \begin{bmatrix} u_0,\ldots,u_{n-2},u_{n-1} \\ x_1,\ldots,x_{n-1},x \end{bmatrix} u_n\Bigg]\right)\begin{bmatrix} u_0,\ldots,u_{n-1},u_n \\ x_1,\ldots,x_n\ \ ,x \end{bmatrix} f\Bigg] +$$

$$+\left(\begin{bmatrix} u_o,\dots,u_{n-2},u_{n-1} \\ x_1,\dots,x_{n-1},x \end{bmatrix} u_n - \begin{bmatrix} u_o,\dots,u_{n-1} \\ x_o,\dots,x_{n-1} \end{bmatrix} u_n\right)\begin{bmatrix} u_o,\dots,u_{n-1},u_n \\ x_o,\dots,x_{n-1},x \end{bmatrix} f =$$

$$= (\alpha-\beta)\begin{bmatrix} u_o,\dots,u_{n-1},u_n \\ x_1,\dots,x_n\ \ ,x \end{bmatrix} f + (\beta-\gamma)\begin{bmatrix} u_o,\dots,u_{n-1},u_n \\ x_o,\dots,x_{n-1},x \end{bmatrix} f, \quad M = \alpha-\gamma.$$

Hence

$$\begin{bmatrix} u_o,\dots,u_n \\ x_o,\dots,x_n \end{bmatrix} f = \frac{\alpha-\beta}{\alpha-\gamma}\begin{bmatrix} u_o,\dots,u_{n-1},u_n \\ x_1,\dots,x_n\ \ ,x \end{bmatrix} f + \frac{\beta-\gamma}{\alpha-\gamma}\begin{bmatrix} u_o,\dots,u_{n-1},u_n \\ x_o,\dots,x_{n-1},x \end{bmatrix} f .$$

This formula holds for any function defined on I. Let us assume that $f(x_j) = 0$ for $j = 0,\dots,n-1$, $f(x) = 0$ and $f(x_n) = 1$. Then $[x_o,\dots,x_n;f]_L > 0$, $[x_1,\dots,x_n,x;f]_L > 0$ and $[x_o,\dots,x_{n-1},x;f]_L = 0$ whence we obtain $\frac{\alpha-\beta}{\alpha-\gamma} > 0$. Analogously $\frac{\beta-\gamma}{\alpha-\gamma} > 0$ whence by induction we obtain (9).

Remark. Applying the definition of LB-splines (see [7,8,12]) we obtain

$$M(t_{k_o},\dots,t_{k_n};t) = \sum_{j=k_o}^{k_n-n}\alpha_j M(t_j,\dots,t_{j+n};t),$$

where $M(x_o,\dots,x_n;t)$ is the LB-spline defined w.r.t. the partition $\Delta = \{x_o < x_1 < \dots < x_n\}$ and the operator L.

The above theorem was proved for the system $\{t^i\}_{i=o}^n$ by T.Popoviciu in [6] (see also [1,2]).

Proof of the properties of the moduli of smoothness. (P.2) follows from the equality $\Delta_h^L f(t) = \sum_{j=0}^{n} f(t+jh)M_j(t)$ analogously as (6). Applying Theorem 4 we prove the remaining properties reasoning analogously as for the modulus $\omega_n(f,h)$ (see [2,10]).

These properties hold for integral moduli of smoothness as well. To prove them we reason analogously. We have only to apply the Minkowski inequality and some properties of functions from $L_p(I)$.

5. An extension of the H.Whitney theorem. We shall prove the following

Theorem 5. Let $f \in C(I)$ and let P_f be a polynomial w.r.t. the system U interpolating f at the points $t_i = \frac{i}{n-1}$, $i = 0,\dots,n-1$. Then

$$|f(t) - P_f(t)| \leq C_L\,\omega_L\left(f,\frac{1}{n-1}\right),$$

where C_L is a constant depending only on the operator L.

To prove it we need the following lemmas:

Lemma 3. Let $0 = m_o < m_1 < \dots < m_{n-1}$ $(m_{n-1} > n)$ be given integers. Then for any integer $s \in (m_o, m_{n-1})$, $y \in I$ and h $(y, y+m_{n-1}h \in I)$ there exist constants a_i and c_j, $i = 0,\dots,m_{n-1} - n = l$, $j = 0,\dots,n-1$ such that for any function $f \in C(I)$

(10) $$f(y+sh) = \sum_{i=0}^{l} a_i \Delta_h^L f(y+ih) + \sum_{j=0}^{n-1} c_j f(y+m_j h).$$

Moreover, if Q is a polynomial w.r.t. the system U such that $Q(y+m_j h) = f(y+m_j h)$, $j = 0,\dots,n-1$, then

(11) $$f(y+sh) = \sum_{i=0}^{l} a_i \Delta_h^L f(y+ih) + Q(y+sh)$$

and $\sum_{i=0}^{l} |a_i| \leq a$, $\sum_{j=0}^{n-1} |c_j| \leq c$, where the constants a and c depend only on the system U and the integer s.

<u>Proof.</u> Applying (5) and Theorem 4 we obtain $f(y+sh) = Q(y+sh) + [y+m_j h, j = 0,\dots,n-1; f]_L \cdot W(y+sh) = Q(y+sh) + \sum_{j=0}^{l} \alpha_j \Delta_h^L f(y+jh) \dfrac{W(y+sh)}{(n-1)!h^n}$ where $\sum_{j=0}^{L} \alpha_j = 1$ and $\alpha_j > 0$. Putting $a_j = \alpha_j W(y+sh)/(n-1)!h^n$ we obtain (11). Hence by (6)

$$\sum_{j=0}^{L} |a_j| \leq d|s - m_0| \cdot \ldots \cdot |s - m_{n-1}| = a.$$

Writing Q in the form $Q(y+sh) = \sum_{j=0}^{n-1} f(y+m_j h) W_j(y+sh)$ and putting $c_j = W_j(y+sh)$ we obtain (10).
Further by (8) we obtain

$$\sum_{j=0}^{n-1} |c_j| \leq \beta \sum_{j=0}^{n-1} |l_j(y+sh)| = c.$$

This completes the proof.

Let now $m_k = k\nu$, $k = 0,\dots,n-1$, $\nu \geq 2$, $s = 1$. Applying Lemma 1 we obtain

(12) $$f(y+h) = \sum_{i=0}^{(n-1)\nu-n} a_i \Delta_h^L f(y+ih) + \sum_{j=0}^{n-1} \gamma_j f(y+j\nu h).$$

Since $\gamma_0 = W_0(y+h)$, we have $0 < \gamma_0 < 1$.

<u>Lemma 4.</u> For every $\varepsilon > 0$ there exists ν such that

$$\sigma = |\gamma_1| + \dots + |\gamma_{n-1}| \leq \varepsilon.$$

<u>Proof.</u> Let P be a polynomial w.r.t. the system U satisfying the following conditions: $P(y) = 0$, $P(y+j\nu h) = 1$ for $\gamma_j \geq 0$ and $P(y+j\nu h) = -1$ for $\gamma_j < 0$. Applying (12) we obtain

$$P(y+h) = \sum_{j=1}^{n-1} |\gamma_j|.$$

Writing the polynomial P in the form (4) and applying (8) we obtain $|P(y+h)| \leq \sum_{j=1}^{n-1} |W_j(y+h)| \leq \beta \sum_{j=1}^{n-1} |l_j(y+h)| = \frac{\beta}{\nu}\left(1 + \frac{1}{2} + \dots + \frac{1}{n-1}\right) < \frac{\beta}{\nu}[1 + \ln(n-1)]$. Putting $\nu > \frac{\beta}{\varepsilon}[1 + \ln(n-1)]$ we obtain the lemma.

Proof of Theorem 5. We may assume that $\omega_L(f,\frac{1}{n-1}) = 1$ and $f(\frac{i}{n-1}) = 0$, $i = 0,\ldots,n-1$. Put $S'_k = \left\{x: x = \frac{i}{2^k(n-1)},\ i = 0,\ldots,2^k(n-1)\right\}$, $S_o = S'_o$, $S_k = S'_k \setminus S'_{k-1}$ $(k>0)$. Choose $\nu = 2r$ from Lemma 4 such that $\sigma < 1$ and μ and m such that $m = 2^\mu(n-1) \geqslant 2(n-1)\nu$. Putting $y = 0$, $m_j h = \frac{j}{n-1}$ we conclude from (10) that there exists a constant M such that $|f(t)| \leqslant M$ for $t \in S'_\mu$. We shall prove that $|f(t)| \leqslant \frac{a+M}{1-\sigma}$ for $t \in \bigcup_{k=0} S_k$. This inequality holds for $t \in S_\mu$. Assume that it holds for $t \in S_k$, $k \geqslant \mu$. We shall prove it for $t \in S_{k+1}$. Let $t \in S_{k+1}$, $0 < t \leqslant \frac{1}{2}$. Then $t = \frac{i}{2^{k+1}(n-1)}$ for some i. Put $y = \max\{x \in S_\mu : x < t\}$ and $h = t - y$. Since ν is even, $y < \frac{1}{2}$ and $h < \frac{1}{m}$, then $y + j\nu h \in S'_k$ for $j = 1,\ldots,n-1$. Hence by (12) and the inductive assumption

$$|f(t)| \leqslant a + \sigma\frac{a+M}{1-\sigma} < \frac{a+M}{1-\sigma}.$$

For $t > \frac{1}{2}$ we put $y = \min\{x \in S_\mu : x > t\}$ and $h = t-y$, and we obtain the same inequality analogously. Since $a < d(n-1)!\nu^{n-1}$ (Lemma 1) and $f \in C(I)$ then $|f(t)| \leqslant C_L = \frac{a+M}{1-\sigma}$ and the proof is completed.

Theorem 5 was proved by H.Whitney in [11] for the system $\{t^i\}_{i=o}^n$ and it's new proof was given by B.Sendov in [9].

6. An extension of the Freud - Popov lemma. Let now $\Delta = \{0 = t_{-n+1} = \ldots = t_o < t_1 < \ldots < t_N = \ldots = t_{N+n-1} = 1\}$, $t_j = \frac{j}{N}$, $j = 0,\ldots,N$, L and L^* be the operators defined in the point 2. We have the following

Theorem 6. (see [4,7,8]) For any $f \in C^n(I)$ there exists a spline s w.r.t. the operator L^*L and the partition Δ such that $s(t_j) = f(t_j)$, $j = 0,\ldots,N$, $D^i s(t_k) = D^i f(t_k)$, $k = 0,N$, $i = 1,\ldots,n-1$ and $\|Ls\|_\infty \leqslant C\|Lf\|_\infty$, where C is a constant depending only on the operator L.

Applying (3) and reasoning analogously as in the proof of Theorem 6 we can prove the following

Lemma 5. For any $f \in C(I)$ there exists a spline s_f w.r.t. the operator L^*L and the partition Δ such that $[t_j,\ldots,t_{j+n};s_f]_L = [t_j,\ldots,t_{j+n};f]_L$ for $j = 0,\ldots,N-n$, $[t_i,\ldots,t_{i+n};s_f]_L = [t_o,\ldots,t_n;f]_L$ for $i = -n+1,\ldots,-1$ and $[t_i,\ldots,t_{i+n};s_f]_L = [t_{N-n},\ldots,t_N;f]_L$ for $i = N-n+1,\ldots,N-1$ and

$$\|Ls_f\|_\infty \leqslant C\max\left\{\left|[t_j,\ldots,t_{j+n};f]_L\right|,\ j = 0,\ldots,N-n\right\},$$

where the constant C depends only on the operator L.

Hence for $h = \frac{1}{N}$

(13) $\quad \|Ls_f\|_\infty \leqslant C[(n-1)!h]^{-n}\omega_L(f,h).$

Let $t \in (t_j, t_{j+n-1})$ and let P_f be a polynomial w.r.t. the operator L (the system U) interpolating the function f at the points t_i, $i = j,\dots,j+n-1$. We have

$$|f(t) - s_f(t)| \leqslant |f(t) - P_f(t)| + |P_f(t) - s_f(t)|.$$

Applying Theorem 5 we obtain

$$|f(t) - P_f(t)| \leqslant C_L\omega_L(f,h).$$

To estimate the second factor, we remark that the polynomial P_f interpolates the spline s_f at the points t_i, $i = j,\dots,j+n-1$. Hence by Theorem 2 and (13) we obtain

$$|P_f(t) - s_f(t)| \leqslant CC_U\omega_L(f,h).$$

Putting these inequalities together and $\varepsilon = \frac{1}{N}$ we obtain

<u>Theorem 7.</u> For any $\varepsilon > 0$ and $f \in C(I)$ there exists a function $f_\varepsilon \in C^n(I)$ such that

$$\|f - f_\varepsilon\|_\infty \leqslant C_1\omega_L(f,\varepsilon)$$

and

$$\|Lf_\varepsilon\|_\infty \leqslant C_2\varepsilon^{-n}\omega_L(f,\varepsilon),$$

where the constants C_1 and C_2 depend only on the operator L.

This theorem was first proved by G.Freud and V.A.Popov in [3] for the operator $L = D^n$ in the space $L_p(I)$, $1 \leqslant p \leqslant \infty$.

<u>7. Best approximation by L-splines.</u> Let $L = D^n + \sum_{i=0}^{n-1} a_i(t)D^i$ be a linear differential operator defined on I with the null space N_L. We can reduce the investigation of L-splines to the investigation of Chebyshev splines by means of the following

<u>Theorem 8.</u> (see [4]). For every operator L of the above form there exists $\delta > 0$ such that, for every subinterval $J \subset I$, with the length $|J| < \delta$ the space N_L has a basis $\{u_i^J\}_{i=0}^{n-1}$, which is an ECT-system in the subinterval J.

Applying theorems 6 - 8 and reasoning analogously as for polynomial splines (see [1,2,7,8]) we can prove the following

<u>Theorem 9.</u> For any partition Δ with sufficiently small $\|\Delta\|$ and any function $f \in C(I)$ there exists an L-spline s_f w.r.t. Δ such that

$$\|f - s_f\|_\infty \leqslant C_L\omega_L(f,\|\Delta\|),$$

where C_L is a constant depending only on the operator L.

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