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EXTREMAL PROPERTIES OF CERTAIN SETS OF SPLINES AND THEIR APPLICATIONS

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1.

Let $u(x)$ and $v(x)$ be continuous functions increasing and respectively decreasing on the real line, $u(0) = v(0) = 0$ and strict monotone in some neighbourhood of the origin. With fixed numbers a and b we associate functions

$$w_{-1}(a,b;x) = -\min(u_-(x-b);\ v_-(x-a)),\quad w_{-1}(a;x) = -v_-(x-a)\ ,$$

$$w_1(a,b;x) = \min(u_+(x-a);\ v_+(x-b))\ ,\quad w_1(a;x) = u_+(x-a)\ ,$$

where

$$f_+(t) = \max(f(t);\ 0)\ ,\qquad f_-(t) = \max(-f(t);\ 0).$$

Let S_{rn} be the set of splines

$$s_r(x) = P_{r-1}(x) + \frac{1}{(r-1)!}\int_0^1 s(z,t)((x-t)_+)^{r-1}\,dt\ , \tag{1}$$

where $P_{r-1} \in \mathcal{T}_{r-1}$, $\mathcal{T}_m$ is the set of algebraic polynomials of degree at most m ,

$$s(z,x) = v_+(x-z_1) + \sum_{k=1}^{n-1} w_{(-1)^k}(z_k,z_{k+1};x) + w_{(-1)^n}(z_n;x)\ , \tag{2}$$

the nodes

$$0 = z_0 \leqslant z_1 \leqslant \ldots \leqslant z_n \leqslant z_{n+1} = 1$$

are arbitrary. It is clear that

$$s_r^{(r)}(x) = s(z,x)\ . \tag{3}$$

$\tilde{S}_{rn}$ is a similar set of 1-periodic splines

$$s_r(x) = a_o + \int_o^1 \tilde{s}(z,t) D_r(x-t)dt ,$$

where $a_o \in \pi_o$,

$$\tilde{s}(z,x) = \sum_{k=1}^{n} w_{(-1)^k}(z_k;z_{k+1};x) , \quad \int_o^1 \tilde{s}(z,x)dx = 0 ,$$

$$z_1 \leq z_2 \leq \ldots \leq z_n \leq z_{n+1} = z_1+1,$$

$$D_m(x) = 2^{1-m} \pi^{-m} \sum_{k=1}^{\infty} k^{-m} \cos(2\pi kx - \pi m/2) .$$

2.

Let $C^r(0,1)$ be the space of functions $f(x)$ having r continuous derivatives on $[0,1]$ ($C^o(0,1) = C$), $\tilde{C}^r$ is the analogous space of 1-periodic functions,

$$\| f \| = \max_x /f(x)/ .$$

By $\mu(f)$ we denote the number of sign changes of $f(x)$ on $[0,1]$ (or on a period when f is periodic) and $\nu(f)$ is the number of zeros of $f(x)$ on $[0,1]$ (or on period) with due regard to their multiplicities. The multiplicity $\rho(t,s)$ of zero t of spline $s \in S_{rn}$ is defined in the usual way: $\rho(t,s) = \min\{ i \leq r : s^{(i)}(t) \neq 0 \}$, if $s^{(i)}(t)=0$ $(i=0:r)$ then $\rho(t,s)=2[(r+1)/2]+1$ when s changes sign at t and $\rho(t,s)=2[(r+2)/2]$ in the other case.

In virtue of (3) the following statement holds true.

LEMMA 1. For every spline $s \in S_{rn}$ $(s \in \tilde{S}_{rn})$

$$\mu(s^{(j)}) \leq n+r-j \qquad (\mu(s^{(j)}) \leq 2[n/2]),$$

$$\nu(s^{(j)}) \leq n+r-j \qquad (\nu(s^{(j)}) \leq 2[(n+1)/2]).$$

LEMMA 2. Let functions u and v satisfy the additional restrictions

$$\mu(u(x)-cu(x-y)) \leq 1 , \quad \mu(v(x)-cv(x-y)) \leq 1 \quad \forall y,c \qquad (4)$$

and s_1, s_2 be functions of the form (2) with n and m nodes respectively. Then for every $c \geq 1$

$$\mu(cs_1 - s_2) \leq n .$$

This follows from (4) and unequalities

$$\mathcal{M}(cu(x-y)-s(x)) \leq 1 \quad , \quad \mathcal{M}(cv(x-y)-s(x)) \leq 1 \quad \forall y,c$$

for every $s \in S_{rn}$.

For set S_{rn} we have the following theorem on zeros.

THEOREM 1. For arbitrary nodes $0 < t_1 < \cdots < t_k < 1$, sets $A,B \in \in N_r = \{0,\ldots,r\}$ and integers $\rho_i \ (1 \leq \rho_i \leq r+2)$,

$$\sum_{i=1}^{k} \rho_i + /A/ + /B/ = n+r ,$$

where /G/ is the number of elements of G, in S_{rn} there exists spline s such that

$$s^{(j)}(t_i)=0 \qquad (i=1:k \ ; \ j=0:\rho_i-1) , \tag{5}$$

$$s^{(j)}(0)=0 \quad (j \in A) , \quad s^{(j)}(1)=0 \quad (j \in B) . \tag{6}$$

Proof. For r=0 the theorem is easy (in this case all $\rho_i=1$ and $z_i=t_i$ in (2)). Let $r \geq 1$, $\rho_i \leq r+1$ $(i=1:k)$. The set of derivatives

$$P(f) = \{ f^{(j)}(t_i) \ (i=1:k, \ j=0:\rho_i-1), \ f^{(a)}(0) \ (a \in A), \ f^{(b)}(1) (b \in B) \}$$

of arbitrary function $f \in C^r(0,1)$ we order

$$P(f) = \{ p_1(f),\ldots,p_{n+r}(f) \}$$

in the following way:

$$p_{j+1}=f^{(j)}(t_1) \ (j=0:\rho_i-1),\ldots, \ p_{j+1+\rho_1+\ldots+\rho_{k-1}}(f)=f^{(j)}(t_k) \ (j=\rho_k-1).$$

Then we number $f^{(a)}(0)$ $(a \in A)$ and $f^{(b)}(1)$ $(b \in B)$.

Now we construct a continuous odd map p from the sphere

$$D = \{ d=(d_1,\ldots,d_{n+1}) , \sum_{i=1}^{n+1} /d_i/ = 1 \}$$

into R_n. Let

$$z_o=0 , \ z_i= \sum_{j=1}^{i} /d_j/ \qquad (i=1:n+1) , \qquad \sigma_i = \operatorname{sgn} d_i ,$$

$$s(d,x) = \sigma_1 v_+(x-z_1) + \sum_{k=1}^{n-1} \sigma_{k+1} w_{(-1)^k}(z_k, z_{k+1};x) + \sigma_{n+1} w_{(-1)^n}(z_n,x)$$

$$s_r(d,x) = P_{r-1}(x) + \frac{1}{(r-1)!} \int_0^1 s(d,t)((x-t)_+)^{r-1}\,dt \quad ,$$

where $P_{r-1} \in \mathcal{T}_{r-1}$ satisfies requirements

$$p_i(s_r) = 0 \qquad (i=1:r) \quad .$$

With each point $d \in D$ we associate the vector $p(d) \in R_n$

$$p(d) = \{ p_{r+1}(s_r), \ldots , p_{r+n}(s_r) \} \quad .$$

It is clear that the map $p\colon D \to R_n$ is continuous and odd. By Borsuk's theorem /1/ there is a point $d_o \in D$ such that $p(d_o)=0$. The function $s_{ro}(x)=s_r(d_o,x)$ satisfies (5)-(6). And $\mu(s_{ro}) \geqslant n$, i.e. $\mu(s(d_o,x)) \geqslant n$. On the other hand, it is obvious that $\mu(s(d_o,x)) \leqslant n$. And for $\rho_i \leqslant r$ (i=1:k) the equality holds if and only if $d_i d_{i+1} > 0$ (i=1:n) . Consequently, either $s_{ro} \in S_{rn}$ or $-s_{rn} \in S_{rn}$. In case $\rho_m > r$ $(1 \leqslant m \leqslant k)$ there should be analogous constructions on $[0,t_m]$ and on $[t_m,1]$. Theorem 1 is proved.

Analogous theorem holds for periodic splines.

THEOREM 2. For all sets of points $t_1 < \cdots < t_k < t_1+1$ and integers $\rho_1, \ldots , \rho_k$ $(1 \leqslant \rho_i \leqslant r+2)$, $\sum_{i=1}^{k} \rho_i = 2m = n$, there is spline $s \in \widetilde{S}_{rn}$ satisfying (5).

Proof. For r=0 the theorem is easy. Let $r \geqslant 1$. The idea of proof is similar to the preceding one. But it has a special difference in construction of the map p. Let $\rho_i \leqslant r+1$ (i = 1:k) and

$$P(f)= \{ f^{(j)}(t_i) \ (i=1:k,\ j=0:\rho_i-1) \} = \{ p_1(f), \ldots , p_n(f) \} \quad .$$

We construct a map p from the sphere

$$D = \{ d=(d_1,\ldots,d_n) \ , \ \sum_{i=1}^{n} /d_i/ =1 \}$$

into R_{n-1} in the following way. With $d \in D$ we associate the values :

$$z_o=0 \ , \ z_i = \sum_{j=1}^{i} /d_j/ \qquad (i=1:n) \ , \qquad \sigma_i = \operatorname{sgn} d_i \quad ,$$

$$s(d,x) = \sum_{k=0}^{n-1} \sigma_{k+1} \, w_{(-1)^k} (z_k, z_{k+1}; x) \quad ,$$

$$s_{rn}(d,x) = \int_0^1 s(d,t)\, D_r(x-t)\, dt \quad ,$$

$$x_d = \inf \left\{ x \in [0,1) : s_{ro}(d,x_d) = s_{ro}(d,x_d+t_2-t_1) \right\} \qquad \text{if} \quad \rho_1 = 1 \ ,$$

$$x_d = \inf \left\{ x \in [0,1) : s'_{ro}(d,x_d) = 0 \right\} \qquad \text{if} \quad \rho_1 > 1 \ ,$$

$$s_r(d,x) = s_{ro}(d,x+x_d-t_1) - s_{ro}(d,x_d) \quad ,$$

$$p(d) = \left\{ p_3(s_r), \dots, p_n(s_r), \int_0^1 s(d,x)\, dx \right\} \ .$$

Now the proof of theorem 2 is completed similar to the preceding one.

REMARK 1. If functions u and v satisfy conditions (4) then there is unique spline $s \in S_{rn}$ with properties (5)-(6); and there are exactly two splines $s_1, s_2 \in S_{rn}$ satisfying (5).

This can be proved using lemma 2 analogous to /2/.

We note that in the case $\rho_i = 2$ (i=1:k) the theorem 2 was proved in /3/.

From the theorems 1 and 2 similar theorems for monosplines (see /4,5,2/ for example) and perfect splines (see /6,7/ for example) follow. It may be obtained if we define u(x)=x , v(x)=-ax and

$$u(x) = -v(x) = \begin{cases} ax \ , & x \in [-1/a,\ 1/a] \\ 1 \ , & x > 1/a \\ -1 \ , & x < -1/a \end{cases}$$

and pass to the limit with $a \to +\infty$.

3.

In this section we obtain the theorems on snakes. For the polynomials those theorems are established by S.Karlin /8/ and extended by other authors (see /9,10/ for example). For the monosplines they are obtained in /11/.

THEOREM 3. Let $f,g \in C$ be functions between which there exists polynomial $Q \in \pi_{r-1}$

$$g(x) < Q(x) < f(x) \quad , \qquad x \in [0,1] \ . \qquad (7)$$

If functions u and v satisfy conditions (4) then there exist exatly two splines $s_1, s_2 \in S_{rn}$ and constants c_1, c_2 $((-1)^r c_1 > 0,\ c_1 c_2 < 0)$ such that

$$g(x) \leq c_i s_i(x) \leq f(x) \quad , \quad x \in [0,1] \qquad (i=1:2) \qquad (8)$$

and for some points

$$0 \leq \xi_1 < \ldots < \xi_{n+r+1} \leq 1 , \qquad 0 \leq \eta_1 < \ldots < \eta_{n+r+1} \leq 1,$$

$$c_1 s_1(\xi_i) = f(\xi_i) , \ c_2 s_2(\eta_i) = g(\eta_i) \qquad (i \text{ odd}) \qquad (9)$$

$$c_1 s_1(\xi_i) = g(\xi_i) , \ c_2 s_2(\eta_i) = f(\eta_i) \qquad (i \text{ even}) \qquad (10)$$

Proof. Consider the simplex

$$D = \left\{ d=(d_0,\ldots,d_{n+r}),\ d_i \geqslant 0,\ \sum_{i=0}^{n+r} d_i = 1 \right\} ,$$

$$t_0=0 , \ t_i = t_i(d) = \sum_{j=0}^{i-1} d_j \qquad (i=1:n+r+1) .$$

$0 \leq \tau_1 < \ldots < \tau_k \leq 1$ are different points of $T_d = \{ t_1,\ldots,t_{n+r} \}$, r_i is a number of knots coinciding with t_i ,

$$\rho_i = \min(r_i,\ r+(3+(-1)^{r_i-r})/2) , \quad m = \sum_{i=1}^{k} \rho_i - r .$$

By theorem 1 and remark 1 with each point $d \in D$ we associate a unique spline $s_d \in S_{rm}$ such that

$$s_d^{(j)}(t_i) = 0 \qquad (i = 1:k,\ j = 0: \rho_i - 1) .$$

According to lemma 1

$$(-1)^{r+i}\, s_d(x) \geqslant 0 , \quad x \in [t_i,t_{i+1}] \qquad (i = 0:n+r) .$$

Therefore the functions

$$p_d(x) = \begin{cases} s_d(x)/(f(x) - Q(x)) , & x \in [t_{i-1},t_i] \quad (i \text{ odd}) \\ s_d(x)/(g(x) - Q(x)) , & x \in [t_{i-1},t_i] \quad (i \text{ even}) , \end{cases}$$

$$q_d(x) = \begin{cases} s_d(x)/(g(x) - Q(x)) , & x \in [t_{i-1},t_i] \quad (i \text{ odd}) \\ s_d(x)/(f(x) - Q(x)) , & x \in [t_{i-1},t_i] \quad (i \text{ even}) \end{cases}$$

are bounded and one of them is nonnegative and the other is nonpositive. Values

$$P_i(d) = \sup \left\{ /p_d(x)/ , \ x \in [t_i,t_{i+1}] \right\} \qquad (i=0:n+r) ,$$

$$Q_i(d) = \sup \left\{ /q_d(x)/ , \ x \in [t_i,t_{i+1}] \right\} \qquad (i=0:n+r)$$

continuously depend on $d \in D$ and

$$P_i(d) = 0 \ , \ Q_i(d) = 0 \iff d_i = 0 \quad (i=0:n+r) \ . \tag{11}$$

Now we show that there exists a point $d^* \in D$ at which

$$P_i(d^*) = P_{i+1}(d^*) = \| p_{d^*} \| \qquad (i=0:n+r) \ .$$

Suppose the contrary: $\forall \ d \in D \ \ \exists \ i=i(d)$ such that $P_i(d) < \| p_d \|$. Then

$$P(d) = \sum_{i=0}^{n+r} (P_i(d) - p(d)) > 0 \ , \quad p(d) = \min_i P_i(d) \ , \tag{12}$$

and the map $h : D \to D$, $h(d) = (h_o(d),\dots,h_{n+r}(d))$,

$h_{j-1}(d) = (P_j(d) - p(d))/P(d) \quad (i=1:n+r+1)$, $P_{n+r+1}(d) = P_o(d)$, is continuous on D. According to the Browder's fixed-point theorem there is $d_o \in D$ such that

$$h(d_o) = d_o = (d_o^o,\dots,d_{n+r}^o) \ ,$$

$$d_{j-1}^o = (P_i(d_o) - p(d_o))/P(d_o) \qquad (i=1:n+r+1) \ . \tag{13}$$

Let $P_l(d_o) = p(d_o)$. Then from (11) and (13) we have

$d_{l-1}^o=0 \Rightarrow \{ P_{l-1}(d_o)=0 \ , \ p(d_o)=0 \} \Rightarrow \ P_{l-2}(d_o)=0 \Rightarrow d_{l-2}^o=0$ etc.

We obtain the equality $P(d_o)=0$. It contradicts to (12)

Thus, there exists a point $d^* \in D$ such that all

$$P_i(d^*) = p(d^*) = \| p_{d^*} \| \neq 0$$

and consequently all $d_i^* > 0$.

In view of continuity of $p_{d^*}(x)$ on intervals (t_i, t_{i+1}) there are points ξ_i $(i=0:n+r)$ at which $/p_{d^*}(\xi_i)/ = \| p_{d^*} \|$. The number $c_1 = 1/p_{d^*}(\xi_o)$ and spline $s_1 = s_{d^*} + P/c_1$ satisfy requirements of the theorem. The second spline is being constructed analogously with the help of function $q_d(x)$.

By analogy to /2, theorem 5.3/ it is easy to establishedthat in S_{rn} there are no other splines satisfying the conditions of the theorem. Theorem 3 is proved.

The following statement can be obtained in the similar way.

THEOREM 4. Let $f,g \in \tilde{C}$ satisfy (7), where $Q \in \mathcal{T}_o$, and u,v have properties (4). Then for arbitrary point t_o there exist exactly two splines $s_1,s_2 \in \tilde{S}_{rn}$ and positive (negative) constants c_1,c_2 satisfying (8),

$$c_1 s_1(t_o) = c_2 s_2(t_o) = Q(t_o) \,,$$

and at some points $\xi_1 < \ldots < \xi_{n+1} < \xi_1 + 1$, $\eta_1 < \ldots < \eta_{n+1} < \eta_1 + 1$, the requirements (9)-(10) are fulfilled.

As in /11/ it is possible to obtain similar theorems if in (8) the symbol of equality occurs at some points.

4.

We introduce in the set of continuous functions the so-called nonsymmetric norm (Minkowski functional). Let $\varphi, \psi \in C$ be strict positive functions. Assume

$$\| f \|_{\varphi,\psi} = \| f_+ \varphi + f_- \psi \| \,.$$

THEOREM 5. Let u and v satisfy conditions (4). Then there is a unique spline $\hat{s} \in S_{rn}$ having n+r+1 alternation points $0 \leqslant x_1 < \ldots < x_{n+r+1} \leqslant 1$ i.e.

$$/\hat{s}(x_i)/ = \| \hat{s} \|_{\varphi,\psi} \qquad \hat{s}(x_i)\, \hat{s}(x_{i+1}) < 0$$

This $\hat{s}$ is the only element of S_{rn} with minimal (φ, ψ)-norm:

$$\| \hat{s} \|_{\varphi,\psi} < \| s \|_{\varphi,\psi} \qquad \forall\, s \in S_{rn} \qquad (s \neq \hat{s}) \,.$$

Proof. According to theorem 3 there exists a unique spline $\hat{s} \in S_{rn}$ and positive constant c satisying (8),(9) (r odd) or (8),(10) (r even) where $f = 1/\varphi$, $g = -1/\psi$:

$$-1/\psi \leqslant c\hat{s} \leqslant 1/\varphi \,.$$

Hence $\hat{s}_+ \varphi \leqslant 1/c$, $\hat{s}_- \psi \leqslant 1/c$ and

$$\| \hat{s} \|_{\varphi,\psi} = \| \hat{s}_+ \varphi + \hat{s}_- \psi \| \leqslant 1/c \,.$$

At n+r+1 points x_i $/\hat{s}(x_i)/ = 1/c$ and $\hat{s}(x_i)\, \hat{s}(x_{i+1}) < 0$.

In analogy with /2, theorem 5.3/ it is possible to prove that $\hat{s}$ has a minimal (φ, ψ)-norm and the uniqueness of the optimal spline. This completes the proof of theorem 5.

The following statement is derived from theorem 4 in a similar way.

THEOREM 6. Let u and v satisfy (4). There exists in $\widetilde{S}_{rn}$ a unique spline $\hat{s}$ with the node $\hat{z}_1 = z$ (z is fixed arbitrary) alternating n times on the period. This $\hat{s}$ is the only element of $\widetilde{S}_{rn}(z) = \{ s \in \widetilde{S}_{rn} : z_1 = z \}$ with minimal (φ, ψ)-norm (in this case we assume that $\varphi, \psi \in \widetilde{C}$).

We apply theorem 5 and 6 to estimates from below of the widths of classes of functions defined by the upper restrictions on the moduli of continuity.

W^rH^ω is the class of functions $f \in C^r(0,1)$ whose moduli of continuity $\omega(f^{(r)},t)$ do not exceed a given concave $\omega(t)$. $\widetilde{W}^rH^\omega$ is the analogous class of 1-periodic functions; and

$$u(x) = v(-x) = \begin{cases} 1/2 \quad (2x), & x \geqslant 0, \\ -1/2 \quad (-2x), & x < 0, \end{cases} \tag{14}$$

The number

$$d_n(M) = \inf_{L_n \subset X} \ \sup_{f \in M} \ \inf_{g \in L_n} \| f - g \|_X$$

is called a n-width in the sense of Kolmogorov of the set $M \subset X$, where L_n denote n-dimensional subspaces of X.

THEOREM 7. The following relations hold for $n \geqslant r$:

$$d_n(W^rH^\omega)_{\varphi,\psi} \geqslant \| s_{rn} \|_{\varphi,\psi} ,$$

where s_{rn} is the spline of minimal (φ,ψ)-norm in the set S_{rn} constructed on the basis of functions (14).

Proof. We follow the reasoning due to N.P.Korneichuk /14, §10.4/. Consider the (n+r+1)-dimensional subspace S_n^r of functions

$$s_r(x) = P_{r-1}(x) + \frac{1}{(r-1)!} \int_0^1 s(z,t)((x-t)_+)^{r-1}\, dt , \tag{15}$$

where $P_{r-1} \in \pi_{r-1}$,

$$s(z,x) = c_o v_+(x-z_1) + \sum_{k=1}^{n-1} c_k w_{(-1)^k}(z_k,z_{k+1},x) + c_n w_{(-1)^n}(z_n,x), \tag{16}$$

z_i are nodes of spline s_{rn} , c_i are real numbers.

Let x_i $(i = 0{:}n+r)$ be alternation points of s_{rn} .

LEMMA 3. If s S_n^r and

$$/s(x_i)/ \leq \| s_{rn} \|_{\varphi,\psi} := d \qquad (i = 0{:}n+r) \tag{17}$$

then the coefficients c_i of the representation (15)-(16) of the spline s satisfy the estimates

$$/c_i/ \leq 1 \qquad (i = 0{:}n)$$

and, consequently, $s \in W^rH^\omega$.

Proof. Assume the contrary: S_n^r contains an element s satisfying (17) but $|c_j| > 1$ for some j. Then

$$|s(x_i)/c_j| < d \qquad (i = 0:n+r)$$

and the difference $h = s_{rn} - s/c_j$ changes sign on (0,1) n+r times at least: $\mathcal{M}(h) \geqslant n+r$. Therefore

$$\mathcal{M}(h^{(r)}) \geqslant n .$$

On the other hand,

$$\mathcal{M}(h^{(r)}) \leqslant n-1$$

because $h^{(r)} \neq 0$ on (z_j, z_{j+1}) . This contrad iction proves lemma 3.

It is not difficult to verify that

$$s_1(x_i) = s_2(x_i) \quad (i = 0:n+r) \quad \Rightarrow \quad s_1 = s_2 \quad .$$

Therefore for any real y_i the set S_n^r contains interpolating element s :

$$s(x_i) = y_i \qquad (i = 0:n+r) \quad .$$

Let L_n be an arbitrary n-dimensional subspace from C(0,1) . In analogy with /14, p.271/ it is not difficult to show that there is an element $s_L \in S_n^r$ such that

$$\max_i |s_L(x_i)| = d$$

and

$$\inf_{l \in L_n} \max_{0 \leqslant i \leqslant n+r} |(s_L(x_i) - l(x_i))_+ \ (x_i) + + (s_L(x_i) - l(x_i))_- \ (x_i)| = d .$$

Hence

$$\inf_{l \in L_n} \| s_L - l \|_{\varphi,\psi} \geqslant d$$

according to lemma 3

$$d_n(W^r H^\omega)_{\varphi,\psi} \geqslant d .$$

Theorem 7 is proved.

Similar theorems hold for the classes

$$W^r H^\omega(A,B) = \{ f \in W^r H^\omega : f^{(i)}(0) = 0 \ (i \in A), \ f^{(j)}(1) = 0 \ (j \in B) \},$$

where A and B are subsets of $N_r = \{ 0,\ldots,r \}$, and for $\widetilde{W}^r H^\omega$.

THEOREM 8. For every r,n = 1,2,... the following estimate is true

$$d_n(\widetilde{W}^r H^\omega)_{\varphi,\psi} \geqslant \| \widetilde{s}_{rn} \|_{\varphi,\psi} ,$$

where s_{rn} is the spline of minimal (φ,ψ)-norm in the set S_{rk}, $k = 2[(n+1)/2]$, constructed on the basis of functions (14).

Those results can be to apply to the problem of optimal recovery of functions.

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