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INTEGRAL REPRESENTATION AND APPROXIMATION OF FUNCTIONS ON A HYPERBOLOID

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1. The Mehler functions. Using the Mehler functions [1,2] $P_{-1/2+i\tau}(\mathrm{ch}\,\theta)=\sqrt{2}\,\pi^{-1}\int_0^\theta(\mathrm{ch}\,\theta-\mathrm{ch}\,s)^{-1/2}\cos\tau s\,ds$, we construct functions $P_{\lambda,\tau}(z)=\frac{\Gamma(1/2+i\tau-\lambda)}{\Gamma(1/2+i\tau+\lambda)}\,d^\lambda/dz^\lambda\,P_{-1/2+i\tau}(z)$, which satisfy the equation $L\,P_{\lambda,\tau}(z)=[(\lambda+1/2)^2+\tau^2]\,P_{\lambda,\tau}(z)$, $L=(z^2-1)^{-\lambda}[(z^2-1)^{\lambda+1}d/dz]$, $L^s=L(L^{s-1})$, $P^{-\lambda}_{-1/2+i\tau}(z)=(z^2-1)^{\lambda}P_{\lambda,\tau}(z)$.

Theorem 1.1. For function $P_{\lambda,\tau}(\mathrm{ch}\,\theta)$ the equality

$$\int_0^h \mathrm{sh}^{-2\lambda-1}t\,dt\int_0^t \mathrm{sh}^{2\lambda+1}\theta\,L\,P_{\lambda,\tau}(\mathrm{ch}\,\theta)\,d\theta=P_{\lambda,\tau}(\mathrm{ch}\,h)-P_{\lambda,\tau}(1)$$

is valid

Proof. If $z=\mathrm{ch}\,\theta$, then $L=\mathrm{sh}^{-2\lambda-1}\theta\,d/d\theta\,(\mathrm{sh}^{2\lambda+1}\theta\,d/d\theta)$.

Consequently $\int_0^t \mathrm{sh}^{2\lambda+1}\theta\,L\,P_{\lambda,\tau}(\mathrm{ch}\,\theta)\,d\theta=\mathrm{sh}^{2\lambda+1}t\,d/dt\,P_{\lambda,\tau}(\mathrm{ch}\,t)$.

The desired equality is obtained by integration.

Theorem 1.2. The equation

$$\frac{\Gamma(\lambda+1)}{\sqrt{\pi}\,\Gamma(\lambda+1/2)}\int_0^\pi P_{\lambda,\tau}(\mathrm{ch}\,\theta)\sin^{2\lambda}\varphi\,d\varphi=\frac{P_{\lambda,\tau}(\mathrm{ch}\,\theta_1)}{P_{\lambda,\tau}(1)}P_{\lambda,\tau}(\mathrm{ch}\,\theta_2),\quad \mathrm{ch}\,\theta=\mathrm{ch}\,\theta_1\,\mathrm{ch}\,\theta_2+\mathrm{sh}\,\theta_1\,\mathrm{sh}\,\theta_2\cos\varphi$$

is valid.

Proof. The expansion

$$P_{\lambda,\tau}(\mathrm{ch}\,\theta)=2^\lambda\Gamma(\lambda)\sum_{j=0}^{\infty}(\lambda+j)\prod_{k=1}^{j}[(\lambda+k-1/2)^2+\tau^2]\,\mathrm{sh}^{-\lambda}\theta_1\,P^{-\lambda-j}_{-1/2+i\tau}(\mathrm{ch}\,\theta_1)\,\mathrm{sh}^{-\lambda}\theta_2\,P^{-\lambda-j}_{-1/2+i\tau}(\mathrm{ch}\,\theta_2)\,C_j^\lambda(\cos\varphi)$$

where $C_j^\lambda(\cos\varphi)$ - the Gegenbauer polinomials is proved in [3].
If we observe that

$$P_{\lambda,\tau}(1)=\frac{1}{2^\lambda\Gamma(\lambda+1)},\quad \int_0^\pi C_0^\lambda(\cos\varphi)\sin^{2\lambda}\varphi\,d\varphi=\frac{\sqrt{\pi}\,\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)},\quad \int_0^\pi C_j^\lambda(\cos\varphi)\sin^{2\lambda}\varphi\,d\varphi=0,\ j=1,2,\dots$$

We receive theorem 1.2.

Theorem 1.3. The equation

$$\int_0^h \mathrm{sh}^{-2\lambda-1}t\,dt\int_0^t \mathrm{sh}^{2\lambda+1}\theta_2\,d\theta_2\left\{\frac{\Gamma(\lambda+1)}{\sqrt{\pi}\,\Gamma(\lambda+1/2)}\int_0^\pi L\,P_{\lambda,\tau}(\mathrm{ch}\,\theta)\sin^{2\lambda}\varphi\,d\varphi\right\}=\left[\frac{P_{\lambda,\tau}(\mathrm{ch}\,h)}{P_{\lambda,\tau}(1)}-1\right]P_{\lambda,\tau}(\mathrm{ch}\,\theta_1)$$

is valid.

Proof. We have from theorem 1.2

$$\frac{\Gamma(\lambda+1)}{\sqrt{\pi}\,\Gamma(\lambda+1/2)}\int_0^\pi L\,P_{\lambda,\tau}(\mathrm{ch}\,\theta)\sin^{2\lambda}\varphi\,d\varphi=\frac{P_{\lambda,\tau}(\mathrm{ch}\,\theta_1)}{P_{\lambda,\tau}(1)}L\,P_{\lambda,\tau}(\mathrm{ch}\,\theta_2)$$

Applying theorem 1.1, we obtain theorem 1.3.

2. Some properties of the functions on a hyperboloid.

Let us define bilinear form $[x,y]=-x_1y_1-\cdots-x_{n-1}y_{n-1}+x_ny_n$ for $x=(x_1,\dots,x_n)\in R_n$, $y=(y_1,\dots,y_n)\in R_n$. Let function f(x) be defined on a hyperboloid $[x,x]=1$, $x_n>0$. We denote $\square=\partial^2/\partial x_1^2+\cdots+\partial^2/\partial x_{n-1}^2-\partial^2/\partial x_n^2$, $\square^s=\square(\square^{s-1})$, $n=2\lambda+3$. Let us consider linear transformation orthogonal with respect to Minkovsky metric $y=Au$, where $A=[a_{ij}]$ is a matrix of transformation. We denote $\|f\|_p=\left(\int_{[x,x]=1}|f(x)|^p dx\right)^{1/p}$ where $1\le p\le\infty$, dx $(n-1)$- dimensional surface element of $[x,x]=1$.

Theorem 2.1. The formula

$$\int_{[y,y]=1} f_1\{[x,y]\} f_2(y)dy=\int_{[u,u]=1} f_1(Mu_n) f_2(Au)du$$

where $M=\{[x,x]\}^{1/2}$, $a_{in}=x_i/M$, $i=1,\dots,n$, $u_n=(1+u_1^2+\cdots+u_{n-1}^2)^{1/2}$ is valid.

Proof. First of all $[y,y]=[u,u]=1$. For Iacobian of transformation A we have $J=(-A_{n1}u_1-\cdots-A_{nn-1}u_n)u_n^{-1}$ where A_{nj} cofactor of the element a_{nj}. As detA=1 we receive $-A_{nj}=a_{nj}$, $j=1,\dots,n-1$, $A_{nn}=a_{nn}$. Therefore $J=y_nu_n^{-1}$. Taking into account the value of Iacobian after change of variables, we obtain theorem 2.1.

Corollary 2.1. If M=1, $f_2=1$, then

$$\int_{[y,y]=1} f_1\{[x,y]\}dy=2\pi^{\frac{n-1}{2}}/\Gamma\left(\frac{n-1}{2}\right)\int_1^\infty f_1(u_n)(u_n^2-1)^{\frac{n-3}{2}}du_n$$

Corollary 2.2. The Young inequality

$$\left\|\int_{[y,y]=1} f_1\{[x,y]\}f_2(y)dy\right\|_r\le\left(2\pi^{\frac{n-1}{2}}/\Gamma\left(\frac{n-1}{2}\right)\int_1^\infty|f_1(u_n)|^q(u_n^2-1)^{\frac{n-3}{2}}du_n\right)^{1/q}\|f_2\|_p$$

$1\le p\le q\le\infty$, $1/r=1/q-1/p\ge 0$ is valid.

Corollary 2.3. Inequality

$$\left\|\int_{[y,y]=1} f_1\{[x,y]\}f_2(y)dy\right\|_p\le 2\pi^{\frac{n-1}{2}}/\Gamma\left(\frac{n-1}{2}\right)\int_1^\infty|f_1(u_n)|(u_n^2-1)^{\frac{n-3}{2}}du_n\,\|f_2\|_p$$

is valid.

Corollary 2.1 is obtained from theorem 2.1 with the help of the Hölder inequality. Corollary 2.3 is obtained from theorem 2.1 with the help of the Minkovsky inequality. If we place the pole of hyperboloid $[x,x]=1$ to the point x of the surface $[x,y]=\operatorname{ch}h$, then from theorem 2.1 we receive parametric equation of surface:

$$y_1=\operatorname{sh}h\sin\theta_{n-2}\cdots\sin\theta_2\sin\theta_1,\ y_2=\operatorname{sh}h\sin\theta_{n-2}\cdots\sin\theta_2\cos\theta_1,\dots,y_{n-1}=\operatorname{sh}h\cos\theta_{n-2}$$

$y_n=\operatorname{ch}h$, $0\le h<\infty$, $0\le\theta_1<2\pi$, $0\le\theta_k\le\pi$, $1<k<n-2$. Thus the surface $[x,y]=\operatorname{ch}h$ is $(n-1)$ - dimensional sphere of radius shh with the centre at the

point X. The area of a surface is equal to $\Omega_\lambda(h)=2\pi^{\lambda+3/2}\,\mathrm{sh}^{2\lambda+1}h/\Gamma(\lambda+3/2)$.
The average function of order K on the hyperboloid $[x,x]=1$ is defined by equation

$$\square^s f_{h_1,\dots,h_K}(x)=\frac{1}{\Omega_\lambda(h_K)}\int_{[x,y]=\mathrm{ch}\,h_K}\square^s f_{h_1,\dots,h_{K-1}}(y)\,dy$$

If $h_1=\dots=h_K=h$, we write $\square^s f_{h_1,\dots,h_K}(x)=\square^s f_{h,K}(x)$, $\square^0 f_{h,0}(x)=f(x)$, $f_{h,1}(x)=f_h(x)$.
We define the binomial difference and modulas of continuity

$$\Delta_h\square^s f(x)=\square^s f_h(x)-\square^s f(x),\quad \Delta_h^K\square^s f(x)=\Delta_h(\Delta_h^{K-1}\square^s f(x)),$$

$$\omega_K(\square^s f,t)_p=\sup_{0<h\le t}\|\Delta_h^K\square^s f(x)\|_p,\quad \omega_1(\square^s f,t)_p=\omega(\square^s f,t)_p.$$

Further we define the generalized wave operator

$$\square_h f(x)=(\lambda+1)\,\mathrm{sh}^{-2}\frac{h}{2}\,[f_h(x)-f(x)]$$

From Taylor expansion it follows that $\lim\limits_{h\to 0}\square_h f(x)=\square f(x)$

Theorem 2.2. For function $P_{\lambda,\tau}\{[y,z]\}$ the expansion

$$P_{\lambda,\tau}\{[y,z]\}=2^\lambda\Gamma(\lambda)\sum_{j=0}^{\infty}(\lambda+j)\prod_{K=1}^{j}[(\lambda+K-1/2)^2+\tau^2]\{[x,y]^2-1\}^{-\lambda/2}P^{-\lambda-j}_{-1/2+i\tau}\{[x,y]\}\times$$

$$\times\{[x,z]^2-1\}^{-\lambda/2}P^{-\lambda-j}_{-1/2+i\tau}\{[x,z]\}C_j^\lambda(\cos\varphi)$$ is valid.

Proof. From the corresponding expression for the function $P_{\lambda,\tau}(\mathrm{ch}\,\theta)$, $[y,z]=[x,y][x,z]+\{[x,y]^2-1\}^{1/2}\{[x,z]^2-1\}^{1/2}\cos\varphi$
connecting the elements of the triangle on a hyperboloid
We obtain the expansion.

Theorem 2.3. The average function of order K for $P_{\lambda,\tau}\{[x,z]\}$ is equal to $P_{\lambda,\tau}\{[x,z]\}_{h,K}=\dfrac{P^K_{\lambda,\tau}(\mathrm{ch}\,h)}{P^K_{\lambda,\tau}(1)}P_{\lambda,\tau}\{[x,z]\}$

Proof. Starting from theorem 2.2 and applying theorem 2.1, we obtain theorem 2.3.

Theorem 2.4. Integral representation

$$J_h^K\square^K P_{\lambda,\tau}\{[x,y]\}_{h,K}=\left[\frac{P_{\lambda,\tau}(\mathrm{ch}\,h)}{P_{\lambda,\tau}(1)}-1\right]^K P_{\lambda,\tau}\{[x,y]\}$$

where $J_h^K=J_h(J_h^{K-1})$, $J_h=\int_0^h \mathrm{sh}_t^{-2\lambda-1}dt\int_0^t \mathrm{sh}^{2\lambda+1}u\,du$ is valid.

Proof. The wave operator becomes the operator L, when the point of differentiation is placed into the pole of a hyperboloid. Therefore starting from theorem 2.3 and applying theorem 1.1 and 1.3 we obtain theorem 2.4.

Theorem 2.5. For function f(x) defined on a hyperboloid integral representation

$$f(x)=\int_0^\infty dm_\lambda(\tau)\int_{[y,y]=1}P_{\lambda,\tau}\{[x,y]\}f(y)\,dy,\quad dm_\lambda(\tau)=(2\pi)^{-\lambda-1}\tau\,\mathrm{th}\,\pi\tau\prod_{K=1}^{\lambda}[(K-1/2)^2+\tau^2]$$

is valid.

Proof. Let $x=(x^{n-1}, x_n)$, $x^{n-1}=(x_1,\dots,x_{n-1})$. Function $f(x)=f(x^{n-1}, x_n)$ have the following expansion

$$f(x)=\sum_{j=0}^{\infty}\sum_{l=1}^{H} a_{l,j}(x_n)Y_{l,j}(x^{n-1}),\quad a_{l,j}(x_n)=\int_{(y^{n-1},y^{n-1})=1} f(y^{n-1}, x_n)Y_{l,j}(y^{n-1})\,dy^{n-1},$$

$$(x^{n-1}, y^{n-1})=x_1y_1+\dots+x_{n-1}y_{n-1},\quad H=(2n+j-2)(n+j-3)!/n!(i-2)!,$$

$$C_j^{\lambda}\{(x^{n-1}, y^{n-1})\}=\frac{2\pi^{\lambda+1}}{\Gamma(\lambda)(\lambda+j)}\sum_{l=1}^{H} Y_{l,j}(x^{n-1})Y_{l,j}(y^{n-1}).$$

Let us define the analogue of Mehler transform:

$$a_{l,j}(x_n)=(2\pi)^{\lambda+j+1}\int_0^{\infty}\tilde{a}_{l,j}(\tau)(x_n^2-1)^{-\lambda/2}P^{-\lambda-j}_{-1/2+i\tau}(x_n)\,dm_{\lambda+j}(\tau)$$

$$\tilde{a}_{l,j}(\tau)=\int_1^{\infty}a_{l,j}(y_n)(y_n^2-1)^{\lambda/2}P^{-\lambda-j}_{-1/2+i\tau}(y_n)\,dy_n$$

From theorem 2.2. With the help of equation

$$\int_{[y,y]=1} f(y)\,dy=\int_1^{\infty}dy_n\int_{(y^{n-1},y^{n-1})=1} f(y^{n-1}, y_n)(y_n^2-1)^{\lambda}\,dy^{n-1}$$

We receive theorem 2.5.

Corollary 2.1. Integral representation of binomial difference $\Delta^K_n f(x)=\gamma^K_h\,\square^K f_{h,K}(x)$.

Theorem 2.6. Inequality $\|\Delta^K_n f(x)\|_p\le h^{2K}\|\Delta^K f(x)\|_p$ is valid.

Proof. The average may be written in the following way

$$\square^K f_{h,K}(x)=\frac{1}{\Omega_{\lambda}(h)}\int_{(y^{n-1},y^{n-1})=sh^2h}\square^K f_{h,K-1}(x^{n-1}+y^{n-1}, x_n+chh)\,dy^{n-1}$$

Using corollary 2.1 generalized Minkovsky inequality and induction, we obtain theorem 2.6.

Theorem 2.7. The average function of order K for f(x) satisfes inequality $\|f_{h,K}(x)\|_p\le\|f(x)\|_p$.

Proof. Using representation

$$f_{h,K}(x)=\frac{1}{\Omega_{\lambda}(h)}\int_{(x^{n-1},y^{n-1})=sh^2h} f_{h,K-1}(x^{n-1}+y^{n-1}, x_n+chh)\,dy^{n-1}$$

induction and generalized Minkovsky inequality, we obtain theorem 2.7.

Theorem 2.8. For $\lambda>0$ inequality $\omega_K(f,\lambda t)_p\le(\lambda+1)^{2K}\omega_K(f,t)_p$ is valid.

Proof. If $m\in N$, then

$$\Delta^K_{mh}f(x)=\prod_{l=1}^{K}\sum_{j_l=1}^{m^2}\iint_{\bar{\Delta}_{j_l}}\square^K f_{u_1,\dots,u_K}(x)\,d\bar{\Delta}_{j_l},\quad d\bar{\Delta}_{j_l}=sh^{-2\lambda-1}t\,sh^{2\lambda+1}u_l\,du_l\,dt$$

where $\bar{\Delta}_{j_l}$ is isosceles right triangle with a leg h. From Minkovsky inequality we have $\|\Delta^K_{mh}f(x)\|_p\le m^{2K}\|\Delta^K_h f(x)\|_p$. As $\omega_K(f,t)_p$ increase and that $m\le\lambda<m+1$ we obtain theorem 2.8.

3. Inequalities of Jackson, Bernstein and Zygmund for functions on a hyperboloid. Starting from the representation of theorem 2.5, we construct the function $g_{\sigma}(x)=\int_0^{\sigma}dm_{\lambda}(\tau)\int_{[y,y]=1}P_{\lambda\tau}\{[x,y]\}g_{\sigma}(y)\,dy$

Theorem 3.1. For $g_\sigma(x)$ inequality of Jackson type $\|g_\sigma(x)\|_\infty \leq C\sigma^{\lambda+1}\|g_\sigma(x)\|_2$. Where C does not depend on $\sigma>0$ is valid.

Proof. By virtue of theorem 2.1 we have

$$g_\sigma(x)=\int_0^\sigma dm_\lambda(\tau)\int_1^\infty \mathcal{P}_{\lambda\tau}(u_n)G(u_n)(u_n^2-1)^\lambda du_n,\quad G(u_n)=\int_{(u^{n-1},u^{n-1})=1} g_\sigma(Au)\,du^{n-1}$$

Using inequality $\left(\int_0^\sigma dm_\lambda(\tau)\right)^{1/2}\leq C\sigma^{\lambda+1}$ and applying Cauchy-Buniakowski inequality and Parseval equality for analogue of Mehler transform, we obtain

$$|g_\sigma(x)|\leq\left(\int_0^\sigma dm_\lambda(\tau)\right)^{1/2}\left\{\int_0^\sigma dm_\lambda(\tau)\left(\int_1^\infty \mathcal{P}_{\lambda\tau}(u_n)G(u_n)(u_n^2-1)^\lambda du_n\right)^2\right\}^{1/2}\leq$$
$$\leq C\sigma^{\lambda+1}\left(\int_1^\infty G^2(u_n)(u_n^2-1)^\lambda du_n\right)^{1/2}=C\sigma^{\lambda+1}\|g_\sigma(x)\|_2$$

Theorem 3.2. For $g_\sigma(x)$ inequality $\|g_\sigma(x)\|_q\leq C\sigma^{(2\lambda+2)(1/p-1/q)}\|g_\sigma(x)\|_p$ $1\leq p\leq q\leq\infty$ where does not depend on σ is valid.

Proof. Choosing least even number that $p_c\geq p$. Using theorem 3.1, we have $\max\limits_x|g_\sigma(x)|^{p_c/2}\leq C_1\left(\frac{\sigma p_c}{2}\right)^{\lambda+1}\|g_\sigma(x)\|_{p_c}^{p_c/2}$. Inequality $\|g_\sigma(x)\|_{p_c}^{p_c/2}\leq\max\limits_x|g_\sigma(x)|^{\frac{p_c-p}{2}}\|g_\sigma(x)\|_p^{p/2}$ is valid. From these inequalities we have $\max\limits_x|g_\sigma(x)|\leq C_2\,\sigma^{\frac{2\lambda+2}{p}}\|g_\sigma(x)\|_p$. Let $q>p$. Then $\|g_\sigma(x)\|_q^q\leq\max\limits_x|g_\sigma(x)|^{q-p}\|g_\sigma(x)\|_p^p$. Further inequality $\max\limits_x|g_\sigma(x)|^{q-p}\leq C_2^{q-p}\sigma^{(2\lambda+2)(q/p-1)}\|g_\sigma(x)\|_p^{q-p}$ is valid. From the last two inequalities we obtain theorem 3.2.

Theorem 3.3. For $g_\sigma(x)$ inequalities of Berstein type

$$\|\Box^s g_\sigma(x)\|_p\leq C\sigma^{2s}\|g_\sigma(x)\|_p \quad\text{and}\quad \|D^s g_\sigma(x)\|_p\leq C\sigma^s\|g_\sigma(x)\|_p$$

$D=x_1\partial/\partial x_1+\dots+x_{n-1}\partial/\partial x_{n-1}-x_n\partial/\partial x_n$, $D^s=D(D^{s-1})$, where C does not depend on $\sigma\geq1$ are valid.

Proof. We denote $K_\nu(h)=[\operatorname{ch}(\sigma h)-1]^{-2\nu}\sin^{2\nu}[\operatorname{ch}(\sigma h)-1]$, $[x,y]=\operatorname{ch}h$, $\nu\in N$, $\nu-\lambda>0$. From the integral representation of function $g_\sigma(x)$ it follows that

$$g_\sigma(x)K_\nu(h_1)=\int_0^\sigma dm_\lambda(\tau)\int_{[y,y]=1}\mathcal{P}_{\lambda\tau}\{[x,y]\}g_\sigma(y)K_\nu(h_2)dy,\ [x,u]=\operatorname{ch}h_1,\ [y,u]=\operatorname{ch}h_2$$

If $x=u$, then $K_\nu(h_1)=1$. Therefore

$$g_\sigma(x)=\int_0^\sigma dm_\lambda(\tau)\int_{[y,y]=1}\mathcal{P}_{\lambda\tau}\{[x,y]\}g_\sigma(y)K_\nu(h)dy$$

From $\Box^s\mathcal{P}_{\lambda\tau}\{[x,y]\}=[(\lambda+1/2)^2+\tau^2]^s\mathcal{P}_{\lambda\tau}\{[x,y]\}$ it follows, that

$$\Box^s\{g_\sigma(x)K_\nu(h_1)\}=\int_0^\sigma dm_{\lambda+s}(\tau)\int_{[y,y]=1}\mathcal{P}_{\lambda\tau}\{[x,y]\}g_\sigma(y)K_\nu(h_2)dy$$

If $x=u$, then $\Box^s\{g_\sigma(x)K_\nu(h_1)\}=\Box^s g_\sigma(x)$. Therefore

$$\Box^s g_\sigma(x)=\int_0^\sigma dm_{\lambda+s}(\tau)\int_{[y,y]=1}\mathcal{P}_{\lambda\tau}\{[x,y]\}g_\sigma(y)K_\nu(h)dy$$

Applying theorem 2.1 und Minkovsky inequality, we receive

$$\| \Box^s g_\sigma(x) \|_p \le \int_0^\sigma \int_0^\infty dm_{\lambda+s}(\tau)\, K_\nu(h)\, sh^{2\lambda+1} h\, dh\, d\tau\, \| g_\sigma \|_p$$

We observe that

$$\int_0^\sigma dm_{\lambda+s}(\tau) \le C_1 \sigma^{2\lambda+2s+2}, \quad \int_0^\infty K_\nu(h)\, sh^{2\lambda+1} h\, dh \le C_2 / \sigma^{2\lambda+2}$$

The second inequality it obtained with the help of inequality $\sigma\, sh\, h \le sh\, \sigma h$, $\sigma > 1$ and the substitution $t = ch(\sigma h) - 1$

Constants C_1 and C_2 does not depend on σ. From the last inequalities we obtain the first inequality of theorem 3.3.

The proof of the second inequality is analogous.

Let $x' = (x_1, \dots, x_m)$, $x'' = (x_{m+1}, \dots, x_n)$, $m = 2\lambda_1 + 3$, $n - m = 2\lambda_2 + 3$.

On a Cartesian product of the surfaces $[x', x'] = 1$ and $[x'', x''] = 1$ we define the function $g_{\sigma_1, \sigma_2}(x', x'')$ by analogy with $g_\sigma(x)$.

We denote $A_\sigma(f)_p = \inf_{g_\sigma} \| f(x) - g_\sigma(x) \|_p$,

$$\| g_{\sigma_1,\sigma_2} \|_{p,x''} = \Big(\int_{[x'',x'']=1} | g_{\sigma_1,\sigma_2}(x', x'') |^p dx' \Big)^{1/p}, \quad \| g_{\sigma_1,\sigma_2} \|_p = \Big(\int_{[x'',x'']=1} \int_{[x',x']=1} | g_{\sigma_1,\sigma_2}(x', x'') |^p dx' dx'' \Big)^{1/p}$$

<u>Theorem 3.4</u>. Inequality of different dimensions

$$\| g_{\sigma_1,\sigma_2} \|_{p,x''} \le C \sigma_2^{\frac{2\lambda_2+2}{p}} \| g_{\sigma_1,\sigma_2} \|_p$$

where constant C does not depend on σ_2 is valid.

<u>Proof</u>. The equations

$$g_{\sigma_1,\sigma_2}(x'; x'') = \int_{[u'',u'']=1} g_{\sigma_1,\sigma_2}(x', u'')\, S_{\sigma_2}\{[x'', u'']\}\, du'', \quad S_{\sigma_2}\{[x'', u'']\} = \int_0^{\sigma_2} \mathcal{P}_{\lambda_2,\tau}\{[x''; u'']\}\, dm_{\lambda_2}(\tau)$$

are valid. If we fulfill calculations analogous those of in theorems 3.1 and 3.2 we shall receive

$$| g_{\sigma_1,\sigma_2}(x', x'') | \le C \sigma_2^{\frac{2\lambda_2+2}{p}} \| g_{\sigma_1,\sigma_2} \|_{p,x''}$$

Integrating last inequality with respect to x'' we obtain theorem 3.4.

Let

$$d_\nu(h) = \gamma_\sigma^{-1} (\sigma\, ch\, h)^{-2\nu} \sin^{2\nu}(\sigma\, ch\, h)\, sh^{2\lambda+1} h, \quad \gamma_\sigma = \frac{2\pi^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty (\sigma\, ch\, h)^{-2\nu} \sin^{2\nu}(\sigma\, ch\, h)\, sh^{2\lambda+1} h\, dh$$

<u>Theorem 3.5</u>. Inequalities

$$\int_0^\infty h^\alpha d_\nu(h)\, dh \le C_1 / \sigma^\alpha, \quad A_\sigma(f)_p \le C_2 / \sigma^2 A_{\sigma-2}(\Box f)_p \quad 2\nu - 2\lambda - \alpha > 1$$

where constants C_1 and C_2 does not depend on σ, are valid.

<u>Proof</u>. First inequality of theorem 3.5 is obtained with the help of change of variable $u = \sigma\, ch\, h$ and inequality $\operatorname{arc} ch\, t \le 2t$, $t \ge 1$.

The second inequality follows from equation

$$f(x) - g_\sigma(x) = \int_0^\infty [f(x) - f_h(x)]\, d_\nu(h)\, dh$$

Minkovsky inequality, theorem 2.6 and the first inequality of theorem 3.5.

Theorem 3.6. Let $f(x)\in L_p$ and $\square^s f(x)\in L_p$. Then there exists function $g_\sigma(x)$ such that for any σ inequality

$$A_\sigma(f)_p \le c/\sigma^{2s}\,\omega_k(\square^s f, 1/\sigma)_p$$

where the constant C not depending on σ, is valid.

Proof. In this theorem the typical cases are k=1 and k=2. With the help of the generalized Minkovsky inequality from equality

$$\square^s f(x) - \square^s g_\sigma(x) = \int_0^\infty [f(x) - f_h(x)]\, d_v(h)dh$$

We receive

$$A_{\sigma-2s}(\square^s f)_p \le \int_0^\infty \omega(\square^s f, h)\, d_v(h)dh$$

From this inequality and theorem 3.5 we receive

$$A_{\sigma-2s}(\square^s f)_p \le \omega(\square^s f, 1/\sigma)_p \int_0^\infty (\sigma h+1)^2 d_v(h)dh \le c_1\omega(\square^s f, 1/\sigma)_p$$

From the last inequality and theorem 3.5 we obtain theorem 3.6 for k=1. By virtue of theorem 2.5 we have

$$\square^s f_{h,2}(x) = \int_0^\infty dm_\lambda(\tau)\frac{P^2_{\lambda\tau}(\operatorname{ch} h)}{P^2_{\lambda\tau}(1)}\int_{[y,y]=1} P_{\lambda\tau}\{[x,y]\}\square^s f(y)dy$$

Let

$$\square^s \tilde f(x) = \int_0^\infty dm_\lambda(\tau)\int_0^\infty \frac{P^2_{\lambda\tau}(\operatorname{ch} h)}{P^2_{\lambda\tau}(1)}\, d_v(h)dh\int_{[y,y]=1} P_{\lambda\tau}\{[x,y]\}\square^s f(y)dy$$

From [4] it follows that $P^2_{\lambda\tau}(\operatorname{ch} h)/P^2_{\lambda\tau}(1)$ is asymptotically $1/\tau^{4\lambda-1}$ as $\tau\to\infty$. Therefore $\square^s\tilde f(x)$ has greater smoothness than $\square^s f(x)$. Hence $A_\sigma(\square^s f)_p$ and $A_\sigma(\square^s f+\square^s\tilde f)_p$ have the same order as $\sigma\to\infty$. From equality

$$\square^s\tilde f(x) - 2\square^s g_\sigma(x) + \square^s f(x) = \int_0^\infty \Delta_h^2\square^s f(x)\, d_v(h)dh$$

it follows that

$$A_{\sigma-2s}(\square^s f)_p \le c_1\int_0^\infty \omega_2(\square^s f, h)\, d_v(h)dh \le c_2\omega_2(\square^s f, 1/\sigma)_p\int_0^\infty (\sigma h+1)^4 d_v(h)dh \le c_3\omega_2(\square^s f, 1/\sigma)_p$$

From the last inequality and theorem 3.5 we obtain theorem 3.6 for k=2.

Theorem 3.7. Let for fixed $\tau>0$ for arbitrary $\sigma>0$ inequality $A_\sigma(f)_p\le c/\sigma^{\tau}$ is valid. Then $\square^s f(x)\in L_p$ and inequality $\omega_k(\square^s f, h)_p \le Mh^{\tau-s}$, $k>\tau-s$ is valid. The constant C does not depend on σ and constant M does not depend on $h>0$

Proof. We denote $Q_0(x)=g_{\sigma_0}(x)$, $Q_i(x)=g_{\sigma_i}(x)-g_{\sigma_{i-1}}(x)$,

$\sigma_i = 2^i\sigma_0$, $\sigma_0>0$, $i=1,2,\dots$ From conditions of theorem 3.7 we have $\|Q_i(x)\|_p \le \|g_{\sigma_i}(x)-f(x)\|_p + \|f(x)-g_{\sigma_{i-1}}(x)\|_p \le c_1/2^{i\tau}$

From theorem 3.3 it follows that $\|\square^s Q_i(x)\|_p \le c_2/2^{i(\tau-s)}$

Therefore the series

$$\square^s f(x) = \sum_{i=0}^\infty \square^s Q_i(x)$$

converges in the L_p.

Further

$$\Delta_h^k \square^s f(x) = \sum_{i=0}^{N} \Delta_h^k \square^s Q_i(x) + \sum_{i=N+1}^{\infty} \Delta_h^k \square^s Q_i(x)$$

where integer N satisfies inequalities $1/2^{N+1} < h \le 1/2^N$.

From the Minkovsky inequality and theorem 2.6 it follows that

$$\|\Delta_h^k \square^s f(x)\|_p \le h^{2k} \sum_{i=0}^{N} \|\square^s Q_i(x)\|_p + 2^k \sum_{i=N+1}^{\infty} \|\square^s Q_i(x)\|_p = I_1 + I_2$$

Therefore

$$I_1 \le h^{2k} c_2 \sum_{i=0}^{N} 2^{ik - i(z-s)} \le c_3 h^{z-s}$$

Further

$$I_2 \le 2^k c_2 \sum_{i=N+1}^{\infty} 2^{-i(z-s)} \le c_4 h^{z-s}$$

Commbining bounds for I_1 and I_2 we obtain theorem 3.7.

<u>Remark</u>. The results, received in this paper, with slight modification are valid for Cartesian product of hyperboloids of different dimensions [5,6]. The special case of theorem 3.6 and 3.7 is the case of half-line considered in [5]. The theorems for half-line are analogues of theorems for segment, which are special cases theorems for sphere received in [7].

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