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ON THE RATE OF APPROXIMATION OF FUNCTIONS BY MULTIPLE ORTHOGONAL SERIES

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1. Introduction

Let (X,F,μ) be a positive measure space, $\Phi = \{\varphi_{ik}(x): i,k=1,2,\ldots\}$ an orthonormal system on X and Ω the set of all such systems. We will consider the double orthogonal series

$$(1.1)\qquad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}\varphi_{ik}(x),$$

where $\{a_{ik}: i,k = 1,2,\ldots\}$ is a double sequence of real numbers for which

$$(1.2)\qquad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 < \infty .$$

By the Riesz-Fischer theorem there exists a function $f(x)\in L^2=L^2(X,F,\mu)$ such that the rectangular partial sums

$$(1.3)\qquad s_{mn}(x) = \sum_{i=1}^{m}\sum_{k=1}^{n} a_{ik}\varphi_{ik}(x) \qquad (m,n = 1,2,\ldots)$$

of series (1.1) converge to $f(x)$ in L^2-metric:

$$\lim_{m,n\to\infty} \int_X [f(x)-s_{mn}(x)]^2 d\mu(x) = 0.$$

We consider also the so-called $(C,1,1)$-means $\sigma_{mn}(x)$ of series (1.1) defined by

$$(1.4)\qquad \sigma_{mn}(x) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n} s_{ik}(x), \quad (m,n = 1,2,\ldots).$$

It is well known that condition (1.2) does not ensure the pointwise convergence of the sequences (1.3) and (1.4) to $f(x)$. The extension of the Menšov-Rademacher theorem proved by a number of authors (see, e.g. 1, 12 etc.) gives the sufficient condition for a.e. convergence of partial sums (1.3): if

$$(1.5)\qquad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 \log^2(i+1)\log^2(k+1) < \infty$$

then for every $\varphi\in\Omega$

$$s_{mn}(x) \to f(x) \text{ a.e. as } m,n\to\infty$$

and there exists a function $F(x)\in L^2$ such that

$$\sup_{m,n\geq 1} |s_{mn}(x)| \leq F(x) \text{ a.e.}$$

Here and in the sequel we make the following convention. Given a double sequence $\{f_{mn}(x)\}$ of functions in L^2, we write $f_{mn}(x) = o_x\{1\}$ a.e. as $m,n\to\infty$ if $f_{mn}(x) \to 0$ a.e. as $m,n\to\infty$ and, in addition, there exists a function $F(x)\in L^2$ such that

$$\sup_{m,n} |f_{m,n}(x)| \leq F(x) \text{ a.e.}$$

In this paper the logarithms are to the base 2.

The test ensuring the a.e. (C,1,1)-summability (see [6]) reads as follows: if

$$(1.6)\qquad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 \log^2\log(i+3)\log^2\log(k+3) < \infty$$

then for every $\varphi\in\Omega$

$$\sigma_{mn}(x) \to f(x) \text{ a.e. as } m,n\to\infty$$

and there exists a function $F(x)\in L^2$ such that

$$\sup_{m,n\geq 1} |\sigma_{mn}(x)| \leq F(x) \text{ a.e.}$$

We will study the rate of the a.e. convergence of the sequences (1.3) and (1.4) to $f(x)$ as $m,n\to\infty$ when the logarithms in (1.5) and the double logarithms in (1.6) are replaced by the multipliers $\lambda_1(i)$ and $\lambda_2(k)$ growing more quickly. The results concerning the a.e. (C,1,1)-summability will be deduced from the a.e. convergence of the Riesz means of the partial sums (1.3), the so-called $(R,P_1,P_2,1,1)$-means

$$R_{mn}(x) = \sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{p_1(i-1)}{p_1(m)}\right)\left(1-\frac{p_2(k-1)}{p_2(n)}\right)a_{ik}\varphi_{ik}(x)\quad (m,n=1,2,\ldots)$$

where $\{p_i(n)\}$, $(i=1,2)$ are strictly increasing to infinity sequences of positive numbers with $p_1(o) = p_2(o) = 0$.

Some new estimates of the order of magnitude of $s_{mn}(x)$ will be done also.

Observe that the results established here in the case of single orthogonal series at most due to V.I. Kolyada [10, 11]. In the particular cases see [2,3,4,7,8,9,13,14].

The theorems 8-10 essentially improve and complete one Moricz's result for (C,1,1)-summability (see [5], theorem 1).

Finally, note that all these results may be easy extended to the multiple orthogonal series.

2. The rate of approximation by partial sums.

Theorem 1. Assume that the positive increasing sequences $\lambda_i(n)$, $(i = 1,2)$ are such that $\lambda_i(n)|\log n$ increase and there exist the sequences $\nu(n,i)$, $(i = 1,2)$ with the properties

$$\mu_i(n) = \nu(n-1,i) - \nu(n,i) \geq 2,\quad 1 < p_i \leq \frac{\lambda_i(\nu(n+1,i))}{\lambda_i(\nu(n,i))} \leq q_i.$$

Then

(i) if

$$(2.1)\qquad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2\lambda_1^2(i)\lambda_2^2(k) < \infty$$

for every $\gamma\in\Omega$ the estimate is true as $m,n\to\infty$

$$(2.2)\qquad f(x) - s_{mn}(x) = o_x\left\{\frac{\log\mu_1(q_1(m))}{\lambda_1(m)} + \frac{\log\mu_2(q_2(n))}{\lambda_2(n)}\right\} \text{ a.e.},$$

where $q_i(n)$, $(i = 1,2)$ is defined with the aid of the strictly increasing function $\nu_i(t)$ with $\nu_i(n) = \nu(n,i)$ and its inverse $\nu_i^{-1}(t)$ by $q_i(n) = \nu_i^{-1}(n)$.

(ii) The estimate (2.2) is final in the following sense: the speed in the right-hand side of (2.2) may not be replaced by a speed tending faster to zero.

This notion of the finality we will apply everewhere in the sequel.

The following corollaries of theorem 1 are very simple and clear.

Corollary 1. Let $\lambda_i(n)$, $(i = 1,2)$ be a positive, increasing to infinity sequence such that for some $\alpha_i\in(0,1)$ the sequence

$\lambda_i(n)2^{-n^{\alpha_i}}$ almost decreases. Under condition (2.1) for every $\gamma\in\Omega$

$$f(x) - s_{mn}(x) = o_x\left\{\frac{\log m}{\lambda_1(m)} + \frac{\log n}{\lambda_2(n)}\right\} \text{ a.e. as } m,n\to\infty$$

and this estimate is final.

Corollary 2. Let $\lambda_i(n)$, $(i = 1,2)$ be a positive sequence such that $n^{-\alpha_i}\log\lambda_i(n)$ increases for some $\alpha_i \in (0,1)$ and $n^{-1}\log\lambda_i(n)$ decreases. Then under condition (2.1) we have the estimate for every $\varphi \in \Omega$ as $m,n \to \infty$

$$f(x)-s_{mn}(x) = o_x\left\{\frac{\log(1+\frac{m}{\log\lambda_1(m)})}{\lambda_1(m)} + \frac{\log(1+\frac{n}{\log\lambda_2(n)})}{\lambda_2(n)}\right\} \text{ a.e.}$$

and this estimate is final.

For example, if $\lambda_i(n) = 2^{n/\log^{\alpha_i} n}$ $(\alpha_i>0;\ i = 1,2)$ then for every Ω we have

$$f(x) - s_{mn}(x) = o_x\left\{\frac{\log\log m}{\lambda_1(m)} + \frac{\log\log n}{\lambda_2(n)}\right\} \text{ a.e. as } m,n \to \infty.$$

Note that for lacunary sequences $\lambda_i(n)$, $(i = 1,2)$ under condition (2.1) the estimate is the best possible:

$$f(x) - s_{mn}(x) = O_x\left\{\frac{1}{\lambda_1(m)} + \frac{1}{\lambda_2(n)}\right\} \text{ a.e. as } m,n \to \infty.$$

Using the above results we study the rate of the a.e. convergence of the subsequences of partial sums (1.3).

Theorem 2. Let $\lambda_i(n)$, $(i = 1,2)$ be a positive, increasing to infinity sequence and $\{m_j\}$, $\{n_l\}$, $\{\nu_i(n)\}$, $(i = 1,2)$ strictly increasing sequences of natural numbers $(\nu_i(n+1) - \nu_i(n) \quad 4)$. Denote $\Lambda_1(j) = \lambda_1(m_j)$ $\Lambda_2(l) = \lambda_2(n_l)$. Then

(i) if $\Lambda_i(n)|\log n$, $(i = 1,2)$ increases and

$1 < q_i \leq \Lambda_i(\nu_i(n+1))|\Lambda_i(\nu_i(n)) \leq r_i$, $(i = 1,2)$, condition (2.1)

implies for every Ω the estimate

$$F(x)-s_{m_j n_l}(x) = o_x\left\{\frac{\log\mu_1(q_1(j))}{\lambda_1(m_j)} + \frac{\log\mu_2(q_2(l))}{\lambda_2(n_l)}\right\} \text{ a.e. as } j,l \to \infty,$$

where $\mu_i(n)$ and $q_i(n)$ are defined in theorem 1.

(ii) If $\Lambda_i(\nu_i(n+1)) = O\{\Lambda_i(\nu_i(n))\}$, $(i = 1,2)$ this estimate is final.

Theorem 3. Let $\lambda_i(n)$, $(i = 1,2)$ be a positive, increasing to infinity sequence and $\{m_j\}$, $\{n_l\}$ strictly increasing sequences of natural nubbers such that $\lambda_1(m_j)|\log j$ and $\lambda_2(n_l)|\log l$ increase to infinity as $j,l \to \infty$. Then under condition (2.1) for every $\varphi \in \Omega$ we have the estimate

$$f(x) - s_{m_j n_1}(x) = O_x\{\frac{\log j}{\lambda_1(m_j)} + \frac{\log l}{\lambda_2(n_1)}\} \text{ a.e. as } j,l \to \infty.$$

Remark 1. In conditions of this theorem the logarithms may be replaced by any sequences with the same order of growth.

Theorem 4. Let $\lambda_i(n)$, $(i = 1,2)$ be a positive, increasing to infinity sequence and $\{m_j\}$, $\{n_1\}$ strictly increasing sequences of natural numbers. Denote $\Lambda_1(j) = \lambda_1(m_j)$, $\Lambda_2(l) = \lambda_2(n_1)$ and assume that $\Lambda_i(n)$, $(i = 1,2)$ satisfies one of conditions:

(i) $\Lambda_i(n)|\log n$ increases to infinity and $\Lambda_i(n)2^{-n^{\alpha_i}}$ almost decreases for some $\alpha_i \in (0,1)$;

(ii) $n^{-1}\log\Lambda_i(n)$ decreases and $n^{-\alpha_i}\log\Lambda_i(n)$ increases for some α_i (0,1).

Then under condition (2.1) for every $\gamma \in \Omega$ the estimate is true as $j,l \to \infty$:

$$f(x)-s_{m_j n_1}(x) = O_x\{\frac{1}{\lambda_1(m_j)}\log(1 + \frac{j}{\log\lambda_1(m_j)}) + \frac{1}{\lambda_2(n_1)}\log(1 + \frac{l}{\log\lambda_2(n_1)})\}$$

a.e. and this estimate is final.

3. The order of growth of partial sums

Theorem 5. Let $\{m_j\}$, $\{n_1\}$ be strictly increasing sequences of natural numbers and $\lambda_i(n)$, $(i = 1,2)$ are positive sequences such that $\lambda_1(m_j)$ logj and $\lambda_2(n_1)$ logl tend to zero monotonically as $j,l \to \infty$. Then the condition (2.1) implies the estimate for every $\gamma \in \Omega$:

$$s_{m_j n_1}(x) = O_x\{\log j.\log l|\lambda_1(m_1).\lambda_2(n_1)\} \text{a.e. as } j,l \to \infty$$

This estimate is final if the sequences $\lambda_i(n)$, $(i = 1,2)$ are separated from zero.

Theorem 6. Let $\{m_j\}$, $\{n_1\}$ be strictly increasing sequences of natural numbers and $\lambda_i(n)$, $(i = 1,2)$ are positive, increasing to infinity

sequences satisfying one of conditions:

(i) $\lambda_1(m_j)2^{-j^{\alpha_1}}$, $\lambda_2(n_1)2^{-l^{\alpha_2}}$ for some $\alpha_1,\alpha_2 \in (0,1)$ almost decrease;

(ii) $j^{-1}\log\lambda_1(m_j)$, $l^{-1}\log\lambda_2(n_1)$ decrease and there exist $\alpha_1,\alpha_2 \in (0,1)$ such that $j^{-\alpha_1}\log\lambda_1(m_j)$, $e^{-\alpha_2}\log\lambda_2(n_1)$ increase.

If

$$(3.1) \qquad \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2\lambda_1^{-2}(i)\lambda_2^{-2}(k) < \infty .$$

then for every $\varphi \in \Omega$ the final estimate takes place as $j,l \to \infty$.

$$s_{m_j n_l}(x) = o_x\{\lambda_1(m_j)\lambda_2(n_l)\log(1 + \frac{j}{\log\lambda_1(m_j)})\log(1 + \frac{l}{\log\lambda_2(n_l)})\} \text{ a.e.}$$

Remark 2. See remark 1.

Theorem 7. Let $\{m_j\}$, $\{n_l\}$ be strictly increasing sequences of natural numbers and $\lambda_i(n)$, $(i = 1,2)$ are positive sequences such that

$$\lambda_1(m_{j+1}) \geq q_1\lambda_1(m_j),\ \lambda_2(n_{l+1}) \geq q_2\lambda_2(n_2)\ (q_1,q_2 > 1;\ j,l = 1,2,\ldots).$$

Then under condition (3.1) for every $\varphi \in \Omega$

$$s_{m_j n_l}(x) = o_x\{\lambda_1(m_j)\lambda_2(n_l)\} \text{ a.e. as } j,l \to \infty$$

and this estimate is final.

4. Riesz summability of double orthogonal series.

In this section we assume that either $P_i(n+1) = O(P_i(n))$ or only one of the sequences $\{P_i(n)\}$, $(i = 1,2)$ is lacunaly.

Theorem 8. Let positive sequences $\lambda_i(n)$, $(i = 1,2)$ satisfy following conditions: $\lambda_i(n)|\log\log P_i(n)$ increases, $\lambda_i(n)|P_i(n)\log\log P_i(n)$ tends monotonically to zero. Then:

1) if (2.1) is fulfilled, for every $\varphi \in \Omega$ the estimate is true as $m,n \to \infty$

$$(4.1)\quad f(x) - R_{mn}(\mathbf{x}) = o_x\{\frac{\log\log P_1(m)}{\lambda_1(m)} + \frac{\log\log P_2(n)}{\lambda_2(n)}\}\text{a.e.}$$

2) The estimate (4.1) is final under one of conditions;

(i) $\lambda_i(n)2^{-\log^{\alpha_i} P_i(n)}$, $(i = 1,2)$ decreases for some $\alpha_i \in (0,1)$;

(ii) $\mu_i(n) \equiv P_i(n)|\lambda_i(n)$, $(i = 1,2)$ increases to infinity but $\mu_i(n)2^{-\log^{\alpha_i} p_i(n)}$ decreases for some $\alpha_i \in (0,1)$;

(iii) $P_i(n)|\lambda_i(n)$, $(i = 1,2)$ decreases.

Corollary 3. If

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2 \log\log^2 P_1(i)\log\log^2 P_2(k) < \infty$$

then for every .

$R_{mn}(x) \to f(x)$ a.e. as $m,n \to \infty$.

Theorem 9. Let $\lambda_i(n)$, $(i = 1,2)$ be a positive sequence such that for some α_i, β_i $(0,1)$, $(i = 1,2)$ the following conditions are fulfilled:

(i) $\log\lambda_i(n)/\log^{\alpha_i} p_i(n)$ increases;

(ii) $\log\lambda_i(n)/\log p_i(n)$ decreases;

(iii) $\log\lambda_i(n) \le \beta_i \log p_i(n)$, $(i = 1,2;\ n = 1,2,\dots)$

Then the condition (2.1) implies for every Ω the final estimate

$$(4.2)\quad f(x)-R_{mn}(x) = O_x\{\log\left(\frac{\log p_1(m)}{\log\lambda_1(m)}\right)/\lambda_1(m) + \log\left(\frac{\log p_2(n)}{\log\lambda_2(n)}\right)/\lambda_2(n)\}$$

a.e. as $m,n \to \infty$.

Remark 3. Observe that in case when $\lambda_i(n) = O(2^{\log^{\gamma_i} p_i(n)})$ for some $\gamma_i \in (0,1)$, $(i = 1,2)$ the estimates (4.1) and (4.2) coincide. If for every $\gamma_i \in (0,1)$ we have $2^{\log^{\gamma_i} p_i(n)} = O\{\lambda_i(n)\}$ and $\lambda_i(n) = O\, p_i^{\beta_i}(n)$ for some $\beta_i \in (0,1)$, $(i = 1,2)$, then (4.2) improves (4.1). For $\lambda_i(n) = p_i^{\beta_i}(n)$, $\beta_i \in (0,1)$, $(i = 1,2)$ the right-hand side of (4.2) has the best possible order $O_x\{\lambda_1^{-1}(m) + \lambda_2^{-1}(n)\}$. Further increasing of $\lambda_i(n)$, $(i = 1,2)$ gives the worse estimate as next theorem chows.

Theorem 10. Let $\lambda_i(n)$, $(i = 1,2)$ be a positive, increasing to infinity sequence such that

(i) $\mu_i(n) \equiv p_i(n)/\lambda_i(n)$ increases to infinity;

(ii) $\log\mu_i(n) | \log p_i(n)$ decreases;

(iii) $\log\mu_i(n) | \log^{\gamma_i} p_i(n)$ increases for some $\gamma_i \in (0,1)$,

$(i = 1,2;\ n = 1,2,\dots)$.

The under condition (2.1) for every Ω the estimate is true as $m,n \to \infty$

$$f(x) - R_{m,n}(x) = O_x\left\{\frac{1}{\lambda_1(m)} \log\left(\frac{\log p_1(m)}{\log\frac{p_1(m)}{\lambda_1(m)}}\right) + \frac{1}{\lambda_2(n)}\log\left(\frac{\log p_2(n)}{\log\frac{p_2(n)}{\lambda_2(n)}}\right)\right\} \text{a.e.}$$

and this estimate is final.

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