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BOUNDS FOR EXTENDED LOCAL LIPSCHITZ CONSTANTS

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1. Introduction. In three recent papers [1, 2, 3], the authors, in collaboration with A. Kroo´ [2, 6], have examined the asymptotic behavior of Lipschitz constants. In the current paper the authors sharpen estimates involving certain Lipschitz constants. First, appropriate terminology is introduced and several definitions are given.

Let $I = [-1, 1]$ and suppose $f \in C[I]$, the space of continuous real-valued functions on I endowed with the uniform norm. Denote the set of all polynomials of degree n or less by π_n, and let $B_n(f)$ be the best uniform approximation to f from π_n. Three Lipschitz constants play prominent roles in this paper. Two of these have received considerable attention in the literature [1, 2, 3, 4, 5, 6]. In particular, for $f \in C[I]$ the classical Lipschitz constant $\lambda_n(f)$ is defined to be

(1.1) $\lambda_n(f) = \sup\{\|B_n(f) - B_n(g)\|/\|f - g\| : g \in C[I],\ f \neq g\}$,

and the local Lipschitz constant $\hat{\lambda}_n(f)$ is defined to be

(1.2) $\hat{\lambda}_n(f) =$

$$\lim_{\delta \to 0^+} \sup\{\|B_n(f) - B_n(g)\|/\|f - g\| : g \in C[I],\ 0 < \|f - g\| < \delta\}.$$

Before defining the third Lipschitz constant, additional terminology is needed. For $f \in C[I]$, let

$$e_n(f)(x) = f(x) - B_n(f)(x),\ x \in I.$$

Then the extremal set of the error function $e_n(f)$ is

$$E_n(f) = \{x \in I : |e_n(f)(x)| = \|e_n(f)\|\}.$$

Let $\mathbf{X}_n = \{x_o, x_1, \ldots, x_{n+1}\} \subseteq E_n(f)$ be an alternant of $e_n(f)$ with $x_o < x_1 < \ldots < x_{n+1}$. Now let $\{q_i\}_{i=0}^{n+1} \subseteq \pi_n$ be determined by

$$q_i(x_j) = (-1)^j, \; i=0, 1, \ldots, n+1; \; i \neq j; \; j=0, 1, \ldots, n+1.$$

If $\mathbf{X}_n = E_n(f)$, then $\hat{\lambda}_n(f)$ can be explicitly displayed [1]. In this case

$$(1.3) \quad \hat{\lambda}_n(f) = \Big\| \sum_{i=0}^{n+1} |q_i|/(1 + |q_i(x_i)| \Big\| = \hat{\lambda}_n(\mathbf{X}_n),$$

where the notation $\hat{\lambda}_n(\mathbf{X}_n)$ is used to emphasize that the local Lipschitz constant depends only on $\mathbf{X}_n$ whenever $E_n(f) = \mathbf{X}_n$. A rather natural and interesting companion to the classical and local Lipschitz constants can now be defined. Specifically, the extended local Lipschitz constant (ELLC) is defined to be

$$(1.4) \quad L_n(f) = \inf\{\lambda_n(h) : h \in C[I], \; E_n(h) = E_n(f), \text{ and } \operatorname{sgn} e_n(h)(x) = \sigma \operatorname{sgn} e_n(f)(x), \; x \in E_n(f), \text{ where } \sigma = +1 \text{ or } -1\}.$$

The ELLC has been defined and analyzed in the more general setting of $C[\mathbf{X}]$, where $\mathbf{X}$ is a closed subset of I, [3]. By focusing attention in the current paper to C[I], sharper estimates for $L_n(f)$ will be obtained. However, several results which are proven in [3] will be useful in the current paper. One of those results is a definition that is equivalent to (1.4). In particular, suppose $f \in C[I]$, and $E_n(f) = \mathbf{X}_n$. Then it can be shown that

$$(1.5) \quad L_n(f) = \inf\{\lambda_n(h) : E_n(h) = X_n \text{ and } h(x_i) = (-1)^i, \; i = 0, 1, \ldots, n+1\}.$$

The main objective of the current paper is to prove the theorem below.

THEOREM. Let $f \in C[I]$, and suppose that $E_n(f) = \{x_0, x_1, \ldots, x_{n+1}\} = \mathbf{X}_n$. Then

$$(1.6) \quad \hat{\lambda}_n(\mathbf{X}_n) \leq L_n(f) \leq 4 + 2\,\hat{\lambda}_n(\mathbf{X}_n).$$

The proof of the theorem depends on three lemmas, two of which appear in [3], and on a rather complex construction of a piecewise linear function h_ℓ.

2. The construction. For $\mathbf{X}_n = \{x_0, x_1, \ldots, x_{n+1}\} \subseteq I$, let

$$d_0 = \begin{cases} x_0 + 1 & \text{if } x_0 > -1 \\ 2 & \text{if } x_0 = -1 \end{cases}$$

and

$$d_{n+1} = \begin{cases} 1 - x_{n+1} & \text{if } x_{n+1} < 1 \\ 2 & \text{if } x_{n+1} = 1 \end{cases}$$

Let $d = \min\{d_0, d_{n+1}, (1/3)\min\{x_{i+1} - x_i : i = 0, 1, \ldots, n\}\}$. Next let $\ell_0 = [1/d] + 1$, where $[Z]$ is the greatest integer in Z. For $\ell \geq \ell_0$, define

$$(2.1) \quad \mathbf{U}_\ell = \left(\bigcup_{i=0}^{n+1} \left(x_i - \frac{1}{\ell}, x_i + \frac{1}{\ell}\right)\right) \cap I.$$

By definition, $d \leq (1/3)\min\{x_{i+1} - x_i : i = 0, 1, \ldots, n\}$, and $(1/\ell) < d$. Therefore the intervals $(x_i - \frac{1}{\ell}, x_i + \frac{1}{\ell})$, $i = 0, 1, \ldots, n+1$ are disjoint. Thus for $\ell \geq \ell_0$, $I - \mathbf{U}_\ell$ is nonempty.

We now begin the construction of $h_\ell \in C[I]$. Let $h_\ell(x_i) = (-1)^i$ and $h_\ell(x_i \pm \frac{1}{\ell}) = 0$ for $i = 0, 1, \ldots, n+1$. For $0 \leq i \leq n$, let

$$(2.2) \quad t_i = 2\left[\frac{x_{i+1} - x_i - (2/\ell)}{(4/\ell)}\right] + 1$$

and

(2.3) $s_i = \dfrac{x_{i+1} - x_i - (2/\ell)}{t_i + 1}$.

From (2.2) and (2.3),

(2.4) $s_i < (2/\ell)$, $i = 0, 1, \ldots, n$.

For $j = 0, 1, \ldots, t_i + 1$; $i = 0, 1, \ldots, n$, let

(2.5) $x_{ij} = x_i + \frac{1}{\ell} + js_i$.

From (2.5) we see that
$x_{io} = x_i + \frac{1}{\ell}$ and $x_{i,t_i+1} = x_{i+1} - \frac{1}{\ell}$, $i = 0, 1, \ldots, n$.

Now let

$h_\ell(x_{ij}) = 0$, $j = 0, 1, \ldots, t_i + 1$; $i = 0, 1, \ldots, n$, and

(2.6) $h_\ell\left(\frac{x_{ij} + x_{i,j+1}}{2}\right) = (-1)^{i+j+1}\left(1 - \frac{1}{\sqrt{\ell}}\right)$,

$j = 0, 1, \ldots t_i$; $i = 0, 1, \ldots, n$.

Note for each $i = 0, 1, \ldots, n$, h_ℓ has local extrema with alternating signs at the points

$\{x_i, \frac{x_{io} + x_{i1}}{2}, \ldots, \frac{x_{it_i} + x_{i,t_i+1}}{2}, x_{i+1}\}$.

If $x_o > -1$ let $x_{-1} = -1 - \frac{1}{\ell}$ and extend the above construction to $i = -1$. Similarly, if $x_{n+1} < 1$, let $x_{n+2} = 1 + \frac{1}{\ell}$ and extend the above construction to $i = n + 1$. Finally, let h_ℓ be linear between the points where the function has already been defined.

By construction,

$$E_n(h_\ell) = \{x_o, x_1, \ldots, x_{n+1}\} = \mathbf{X}_n \text{ and } B_n(h_\ell) \equiv 0.$$

Clearly

$$h_\ell \varepsilon \{h : E_n(h) = \mathbf{X}_n \text{ and } h(x_i) = (-1)^i, i=0, 1, \ldots, n+1\}.$$

Therefore if $f \varepsilon C[I]$ and $E_n(f) = \mathbf{X}_n$, then (1.5) implies that

(2.7) $L_n(f) \leq \lambda_n(h_\ell)$.

Let $x \in I$. The construction of h_ℓ insures that there exists a $y \in I$ such that $|x-y| \leq \frac{3}{\ell}$ and $h_\ell(y) = 1 - \frac{1}{\sqrt{\ell}}$, and that there exists a $\bar{y} \in I$ with $|x - \bar{y}| \leq \frac{3}{\ell}$ and $h_\ell(\bar{y}) = -(1 - \frac{1}{\sqrt{\ell}})$.

3. Lemmas. This section consists of three lemmas. The first two are needed in the proof of the third, which in turn facilitates the proof of the theorem. The proofs of the first two lemmas involve modifications of proofs of similar lemmas appearing in [3]. Thus their proofs are omitted here. The set H_ℓ below is used in the first lemma and in the proof of the third.

(3.1) $H_\ell = \{g \in C[I] : e_n(g)$ possesses no alternant $\{y_0, y_1, \ldots, y_{n+1}\}$ with $y_i \in (x_i - \frac{1}{\ell}, x_i + \frac{1}{\ell}) \cap I$ and $\operatorname{sgn} e_n(g)(y_i) = (-1)^i$, $i = 0, 1, \ldots, n + 1\}$,

where again, $\mathbf{X}_n = \{x_0, x_1, \ldots, x_{n+1}\}$.

LEMMA 1. *For $\ell > \ell_0$, let $\beta_\ell = \inf\{\|g - h_\ell\| : g \in H_\ell\}$. Then there exists a $\bar{g}_\ell \in C[I]$ and an $\bar{x}_\ell \in I$ with $\|\bar{g}_\ell - h_\ell\| < \beta_\ell$, $|e_n(\bar{g}_\ell)(\bar{x}_\ell)| \geq \|e_n(\bar{g}_\ell)\| - \frac{1}{\ell}$, and either $\bar{x}_\ell \in I - U_\ell$ or $\bar{x}_\ell \in (x_i - \frac{1}{\ell}, x_i + \frac{1}{\ell}) \cap I$ and $\operatorname{sgn} e_n(\bar{g}_\ell)(\bar{x}_\ell) = (-1)^{i+1}$ for some i, $0 \leq i \leq n + 1$.*

LEMMA 2. *If $g \in C[I]$ is such that $e_n(g)$ possesses an alternant $\{y_0, y_1, \ldots, y_{n+1}\}$ with $y_i \in (x_i - \frac{1}{\ell}, x_i + \frac{1}{\ell}) \cap I$ and $\operatorname{sgn} e_n(g)(y_i) = (-1)^i$, $i = 0, 1, \ldots, n + 1$, then there is a constant K independent of g and ℓ such that*

(3.2) $$\frac{\|B_n(g) - B_n(h_\ell)\|}{\|g - h_\ell\|} \leq (1 + \frac{K}{\ell})\, \hat{\lambda}_n(\mathbf{X}_n).$$

LEMMA 3. *If $g \in C[I]$ and $e_n(g)$ possesses no alternant $\{y_0, y_1, \ldots, y_{n+1}\}$ with $y_i \in (x_i - \frac{1}{\ell}, x_i + \frac{1}{\ell}) \cap I$ and* sgn $e_n(g)(y_i) = (-1)^i$ *for* $i = 0, 1, \ldots, n + 1$, *then*

(3.3) $\quad \|g - h_\ell\| \geq (\frac{1}{\sqrt{\ell}} - \frac{1}{\ell}) / (2(1 + (1 + \frac{K}{\ell}) \hat{\lambda}_n(\mathbf{X}_n)))$, *where the constant* K *is described in Lemma 2*.

Proof: From Lemma 1 there exists a $\bar{g}_\ell$ and $\bar{x}_\ell$ with $\|\bar{g}_\ell - h_\ell\| < \beta_\ell$,

(3.4) $\quad |e_n(\bar{g}_\ell)(\bar{x}_\ell)| \geq \|e_n(\bar{g}_\ell)\| - \frac{1}{\ell}$,

and either $\bar{x}_\ell \in I - \mathbf{U}_\ell$ or $\bar{x}_\ell \in (x_i - \frac{1}{\ell}, x_i + \frac{1}{\ell}) \cap I$ for some i and

(3.5) $\quad \operatorname{sgn} e_n(\bar{g}_\ell)(\bar{x}_\ell) = (-1)^{i+1}$.

Since $\|\bar{g}_\ell - h_\ell\| < \beta_\ell$, the definitions of β_ℓ and $\mathbf{H}_\ell$ imply that $e_n(\bar{g}_\ell)$ has an alternant $\{y_o, y_1, \ldots, y_{n+1}\}$ with $y_i \in (x_i - \frac{1}{\ell}, x_i + \frac{1}{\ell}) \cap I$ and $\operatorname{sgn} e_n(\bar{g}_\ell)(y_i) = (-1)^i$, $i = 0, 1, \ldots n + 1$. Now

$$\begin{aligned}
\|e_n(\bar{g}_\ell)\| &\geq |e_n(\bar{g}_\ell)(x_o)| \\
&\geq |h_\ell(x_o) - B_n(h_\ell)(x_o)| \\
&\quad - |\bar{g}_\ell(x_o) - h_\ell(x_o)| \\
&\quad - |B_n(h_\ell)(x_o) - B_n(\bar{g}_\ell)(x_o)| \\
&\geq 1 - \|\bar{g}_\ell - h_\ell\| - \|B_n(h_\ell) - B_n(\bar{g}_\ell)\|.
\end{aligned}$$

Thus from (3.2) we have that

(3.6) $\quad \|e_n(\bar{g}_\ell)\| \geq 1 - (1 + (1 + \frac{K}{\ell}) \hat{\lambda}_n(\mathbf{X}_n))\|\bar{g}_\ell - h_\ell\|$.

First suppose that $\bar{x}_\ell \in I - \mathbf{U}_\ell$. Then from the construction of h_ℓ we have that $|h_\ell(\bar{x}_\ell)| \leq 1 - \frac{1}{\sqrt{\ell}}$. Therefore

$$|e_n(\bar g_\ell)(\bar x_\ell)| \le |\bar g_\ell(\bar x_\ell) - h_\ell(\bar x_\ell)| + |h_\ell(\bar x_\ell)|$$
$$+|B_n(\bar g_\ell)(\bar x_\ell) - B_n(h_\ell)(\bar x_\ell)|$$
$$\le \|\bar g_\ell - h_\ell\| + 1 - \frac{1}{\sqrt{\ell}} + \| B_n(\bar g_\ell) - B_n(h_\ell)\|.$$

Now (3.2) implies that

$$|e_n(\bar g_\ell)(\bar x_\ell)| \le (1 + (1 + \frac{K}{\ell})\ \hat\lambda_n(X_n))\|\ \bar g_\ell - h_\ell\| + 1 - \frac{1}{\sqrt{\ell}}.$$

Thus (3.4) assures that

$$(3.7) \quad \|e_n(\bar g_\ell)\| \le (1 + (1 + \frac{K}{\ell})\ \hat\lambda_n(X_n))\|\ \bar g_\ell - h_\ell\| + 1 + \frac{1}{\ell} - \frac{1}{\sqrt{\ell}}.$$

Next assume that $\bar x_\ell \in (x_i - \frac{1}{\ell}, x_i + \frac{1}{\ell}) \cap I$ for some $i = 0, 1, \ldots, n+1$. Without loss of generality we may assume that i is even. Then from (3.5), $e_n(\bar g_\ell)(\bar x_\ell) < 0$ and $e_n(h_\ell)(x_i) = h_\ell(x_i) = 1$. Thus from the construction of h_ℓ we have $h_\ell(\bar x_\ell) \ge 0$. If $\bar g_\ell(\bar x_\ell) < 0$, then $|\bar g(\bar x_\ell)| \le |\bar g_\ell(\bar x_\ell) - h_\ell(\bar x_\ell)|$. Therefore

$$|e_n(\bar g_\ell)(\bar x_\ell)| \le |\bar g_\ell(x_\ell)| + |B_n(\bar g_\ell)(\bar x_\ell)|$$
$$\le |\bar g_\ell(\bar x_\ell) - h_\ell(\bar x_\ell)|$$
$$+ |B_n(\bar g_\ell)(\bar x_\ell) - B_n(h_\ell)(\bar x_\ell)|$$
$$\le \|\bar g_\ell - h_\ell\| + \|B_n(\bar g_\ell) - B_n(h_\ell)\|.$$

By using (3.4) and (3.2), we again obtain (3.7). If on the other hand $\bar g_\ell(\bar x_\ell) \ge 0$, then (3.5) implies that $0 \le \bar g_\ell(\bar x_\ell) < B_n(\bar g_\ell)(\bar x_\ell)$. Therefore

$$|e_n(\bar g_\ell)(\bar x_\ell)| \le |B_n(\bar g_\ell)(\bar x_\ell)| \le |\bar g_\ell(\bar x_\ell) - h_\ell(\bar x_\ell)|$$
$$+ |B_n(\bar g_\ell)(\bar x_\ell) - B_n(h_\ell)(\bar x_\ell)|.$$

Thus as above, we once again obtain (3.7). We have shown that (3.7) is valid in all cases. Inequalities (3.6) and (3.7) combine to imply that

$$(3.8) \quad \|\bar{g}_\ell - h_\ell\| > \frac{\frac{1}{\sqrt{\ell}} - \frac{1}{\ell}}{2(1 + (1 + \frac{K}{\ell})\hat{\lambda}_n(\mathbf{X}_n))}.$$

If $g \in C[I]$ and $e_n(g)$ has no alternant $\{y_o, y_1, \ldots, y_{n+1}\}$ where $y_i \in (x_i - \frac{1}{\ell}, x_i + \frac{1}{\ell}) \cap I$ and $\operatorname{sgn} e_n(g)(y_i) = (-1)^i$, $i = 0, 1, \ldots, n + 1$, then (3.8) implies that

$$(3.9) \quad \|g - h_\ell\| > \beta_\ell > \|\bar{g}_\ell - h_\ell\|$$

$$> \frac{\frac{1}{\sqrt{\ell}} - \frac{1}{\ell}}{2(1 + (1 + \frac{K}{\ell})\hat{\lambda}_n(\mathbf{X}_n))},$$

which establishes (3.3).

4. Proof of theorem. Let $g \in C[I]$ with $\|g - h_\ell\| \neq 0$. We first suppose that

$$\|g - h_\ell\| < \frac{\frac{1}{\sqrt{\ell}} - \frac{1}{\ell}}{2(1 + (1 + \frac{K}{\ell})\hat{\lambda}_n(\mathbf{X}_n))}.$$

Then by (3.9) and the definition of β_ℓ, $e_n(g)$ has an alternant $\{y_o, y_1, \ldots, y_{n+1}\}$ with $y_i \in (x_i - \frac{1}{\ell}, x_i + \frac{1}{\ell}) \cap I$ and $\operatorname{sgn} e_n(g)(y_i) = (-1)^i$ for $i = 0, 1, \ldots, n + 1$. In this case Lemma 2 implies that

$$(4.1) \quad \frac{\|B_n(g) - B_n(h_\ell)\|}{\|g - h_\ell\|} < (1 + \frac{K}{\ell})\hat{\lambda}_n(\mathbf{X}_n).$$

Next suppose that

$$(4.2) \quad \|g - h_\ell\| > \frac{\frac{1}{\sqrt{\ell}} - \frac{1}{\ell}}{2(1 + (1 + \frac{K}{\ell})\hat{\lambda}_n(\mathbf{X}_n))}.$$

If $\|g\| > 2$, then

$$(4.3) \quad \frac{\|B_n(g) - B_n(h_\ell)\|}{\|g - h_\ell\|} < \frac{\|B_n(g)\|}{\|g\| - 1} < \frac{2\|g\|}{\|g\| - 1} < 4.$$

If $\|g\| < 2$, choose $\bar{x} \in I$ such that $|B_n(g)(\bar{x})| = \|B_n(g)\|$. Without loss of generality suppose that $B_n(g)(\bar{x}) > 0$. By the construction of h_ℓ, we may select a $\bar{y} \in I$ such that $|\bar{x} - \bar{y}| < \frac{3}{\ell}$

and $h_\ell(\bar{y}) = -(1 - \frac{1}{\sqrt{\ell}})$. Now

$|B_n(g)(\bar{y}) - B_n(g)(\bar{x})| = |\bar{y} - \bar{x}| \; |(B_n(g))'(r)|$ for some r between $\bar{x}$ and $\bar{y}$. Then Markoff's inequality implies that

$$|B_n(g)(\bar{y}) - B_n(g)(\bar{x})| \leq \frac{3n^2}{\ell} \|B_n(g)\| \leq \frac{6n^2}{\ell} \|g\| \leq \frac{12n^2}{\ell}.$$

Thus

$$\|B_n(g) - B_n(h_\ell)\| = \|B_n(g)\| = B_n(g)(\bar{x})$$
$$= B_n(g)(\bar{y}) + B_n(g)(\bar{x}) - B_n(g)(\bar{y})$$
$$\leq B_n(g)(\bar{y}) + \frac{12n^2}{\ell}$$
$$= B_n(g)(\bar{y}) - g(\bar{y}) + g(\bar{y}) - h_\ell(\bar{y}) + h_\ell(\bar{y}) + \frac{12n^2}{\ell}$$
$$\leq \|e_n(g)\| + \|g - h_\ell\| - (1 - \frac{1}{\sqrt{\ell}}) + \frac{12n^2}{\ell}.$$

But $\|e_n(g)\| \leq \|g\| \leq \|g - h_\ell\| + \|h_\ell\| = \|g - h_\ell\| + 1$.

Therefore $\|B_n(g) - B_n(h_\ell)\| \leq 2\,\|g - h_\ell\| + \frac{1}{\sqrt{\ell}} + \frac{12n^2}{\ell}$.

This inequality and the assumption in (4.2) imply that

$$(4.4) \quad \frac{\|B_n(g) - B_n(h_\ell)\|}{\|g - h_\ell\|} \leq 2 + 2\,(1 + (1 + \frac{K}{\ell})\,\hat{\lambda}_n(\mathbf{X}_n))\,\left(\frac{\frac{1}{\sqrt{\ell}} + \frac{12n^2}{\ell}}{\frac{1}{\sqrt{\ell}} - \frac{1}{\ell}}\right).$$

Taken together, (4.1), (4.3), and (4.4) imply that

$$\lambda_n(h_\ell) \leq \max\{(1 + \frac{K}{\ell})\,\hat{\lambda}_n(\mathbf{X}_n),\ 4,$$
$$2 + 2(1 + (1 + \frac{K}{\ell})\,\hat{\lambda}_n(\mathbf{X}_n))\,\left(\frac{1 + \frac{12n^2}{\sqrt{\ell}}}{1 - \frac{1}{\sqrt{\ell}}}\right)\}.$$

From (2.7) we now have that

$$(4.5) \quad L_n(f) \leq \max\,\{(1 + \frac{K}{\ell})\,\hat{\lambda}_n(\mathbf{X}_n),\ 4,$$
$$2 + 2\,(1 + (1 + \frac{K}{\ell})\,\hat{\lambda}_n(\mathbf{X}_n))\,\left(\frac{1 + \frac{12n^2}{\sqrt{\ell}}}{1 - \frac{1}{\sqrt{\ell}}}\right)\}.$$

Letting $\ell \to \infty$ in (4.5) yields

$$(4.6) \quad L_n(f) \le 4 + 2 \hat{\lambda}_n(X_n),$$

the upper bound in (1.6). The proof for the lower bound in (1.6) appears in [3]. Thus the proof of the theorem is complete.

With appropriate modifications in the construction of h_ℓ, the theorem can be established for any closed, dense subset of I. Finally, it would be of interest to find functions $f \in C[I] - \pi_n$ for which $\{\frac{\lambda_n(f)}{L_n(f)}\}_{n=0}^{\infty}$ is a bounded set.

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