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A VARIATIONAL APPROACH TO MONOSPLINES

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1. Introduction. The Fundamental Theorem of Algebra for Monosplines has attracted the attention of many outstanding mathematicians. This includes Schoenberg, Karlin and Schumaker, and Micchelli. (See Micchelli [4] for references)

The Fundamental Theorem of Algebra for Monosplines (with odd multiplicity) may be stated as follows:

Let $N = \overline{m}_0 + \sum_{i=1}^{r} (\overline{m}_i + 1) = \sum_{i=1}^{s} \overline{n}_i$; $\overline{m}_0 \le m$; $\overline{m}_i$ and $\overline{n}_i \le m$;

m_i are odd, $i \neq 0$ (1)

$$\Phi_m(x, v) = \frac{(x-v)_+^{m-1}}{(m-1)!} \; ; \; \Phi_m{}^j(x, v) = \frac{\partial^j}{\partial v^j} \Phi_m(x, v) .$$

Assume (1) holds, and given s distinct zeros $0 < y_1 < y_2 < \cdots < y_s \le 1$ show the existence and uniqueness of r distinct knots $0 = t_0 < t_1 < t_2 < \cdots < t_r \le 1$ and constants a_j and a_{ij} such that

$$M(y) = \int_0^1 \Phi_m(y, t)dt - \sum_{j=0}^{\overline{m}_0 - 1} a_j \Phi_m{}^j(y, t_0) - \sum_{i=1}^{r} \sum_{j=0}^{\overline{m}_i - 1} a_{ij} \Phi_m{}^j(y, t_i) \qquad (2)$$

has a zero of order $\bar{n}_\nu$ at y_ν $\nu = 1, \ldots, s$.

Recently Zhensykbaev [6] by using the Brouwer Fixed Point Theorem, has given a necessary and sufficient condition for the solution of this problem.

We have proven the uniqueness in [1]. In this paper, by using a variational principle we will establish necessary and sufficient conditions equivalent to those of Zhensykbaev for the existence of solutions.

Since any monospline of the form (2) has at most $\bar{m}_0 + \sum_{i=1}^{r} (\bar{m}_i + 1) = N$ zeros if one has N zeros we say it has a full set of zeros.

In regard to (2), we have the following Lemma.

Lemma A. Let $M(y)$ have a full set of zeros in $[0, 1]$. Moreover let $y_g = t_k = a$, $0 < a < 1$. with $\bar{n}_g + \bar{m}_k \geq m + 1$, then $M(y)$ has a full set of zeros in $[0, a]$ with the zero at a of multiplicity $\bar{n}_g - 1$, and a full set of zeros in $[a, 1]$ with a knot of multiplicity $\bar{m}_k$ at a, i.e. under these circumstances equation (1) can be broken up into three equations.

$$\bar{m}_0 + \sum_{i=1}^{k-1} (\bar{m}_i + 1) = \sum_{i=1}^{g-1} \bar{n}_i + (\bar{n}_g - 1) \tag{3}$$

$$\bar{m}_k + \sum_{i=k+1}^{r} (\bar{m}_i + 1) = \sum_{i=g+1}^{s} \bar{n}_i \tag{4}$$

with

$$\bar{m}_k + \bar{n}_g \geq m + 1 . \tag{5}$$

Conversely, if we can solve the fundamental theorem of algebra problem corresponding to (3) on $[0, a]$ and to (4) on $[a, 1]$, with $\bar{m}_k + \bar{n}_g \geq m + 1$, then we can solve the Fundamental Theorem of Algebra problem corresponding to (1) on $[0, 1]$.

We note that if $\overline{m}_k = m$ (odd), and $M(y)$ has a full set of zeros it follows that $t_k = y_g$ for some g (Micchelli [4], Cor. 2). Thus in this case $\overline{m}_k + \overline{n}_g \geq m + 1$ and the above lemma shows we can break $M(y)$ up into two parts, which we can solve separately.

Hence we need only establish the Fundamental Theorem of Algebra for those cases where $\overline{m}_k < m$ all k and (1) cannot be broken up into (3), (4) and (5).

Let $\bar{t}_1, \ldots, \bar{t}_N$ be the sequence obtained from $t_0, \ldots, t_r$, by repeating t_0, $\overline{m}_0$ times and t_i, $\overline{m}_i + 1$ times $i = 1, \ldots, r$, and let $\bar{y}_1, \ldots, \bar{y}_N$ be the sequence obtained from $y_1, \ldots, y_s$ by repeating y_i, $\bar{n}_i$ times $i = 1, \ldots, s$. Our principle result is:

<u>Main Theorem</u>. A necessary and sufficient condition for the Fundamental Theorem of Algebra to hold for $\overline{m}_k < m$, under the proviso that if a is a knot of multiplicity $\overline{m}_k$ and a zero of multiplicity $\bar{n}_g$ then $\overline{m}_k + \bar{n}_g \leq m$ is that the set $\bar{D}$ of all $\bar{t}_i$ such that:

$$\bar{y}_i < \bar{t}_i < \bar{y}_{i+m} \quad i = 1, \ldots, N$$

(where the right side is ignored if $i + m > N$) is non-empty.

We note that Micchelli [4], Cor. 2., established the necessity, (see also Schumaker [5], Theorem 8.44). We devote this paper to establishing the sufficiency.

For latter use, let us introduce the following notation:

$$t = 1 - x;\ t_i = 1 - x_{r+1-i}\quad i = 0, \ldots, r;\ 1 + \bar{m}_i = m_{r+1-i}\quad i = 1, \ldots, r;$$

$$\bar{m}_0 = m_{r+1}\qquad y = 1 - v;\ y_i = 1 - v_{s+1-i},\ n_i = \bar{n}_{s+1-i}\quad i = 1, \ldots, s$$

$$n_0 = 0 \quad g_i = \sum_{j=0}^{i} n_j$$

$$u_{g_{i-1}+j}(x) = \Phi_m^{j-1}(x, v_i) \begin{cases} i = 1, \ldots, s, \\ j = 1, \ldots, n_i, \end{cases} \quad \text{defines } u_1, \ldots, u.$$

Then the Fundamental Theorem of Algebra problem becomes:

$$\text{Let } N = \sum_{i=1}^{s} n_i = m_{r+1} + \sum_{i=1}^{r} m_i; \qquad m_i(\text{even}) \leq m,\ m_{r+1} \leq m$$

Given points $0 < v_1 < v_2 < \cdots < v_s < 1$, find points

$0 < x_1 < x_2 < \cdots < x_r < x_{r+1} = 1$, and constants $\bar{a}_j, \bar{a}_{ij}$ such that

$$\int_0^1 u_\nu(x) = \sum_{j=0}^{m_{r+1}-1} \bar{a}_j\, u_\nu^{j}(1) + \sum_{i=1}^{r} \sum_{j=0}^{m_i-2} \bar{a}_{ij}\, u_\nu^{j}(x_i);\quad \nu = 1, \ldots, N. \tag{6}$$

i.e. the problem can be formulated in terms of finding a quadrature formula.

Motivated by our recent paper [2], we recast the solution of (6) into the following problem.

Given $0 = v_0 < v_1 < \cdots < v_{s+1} = 1$ and $\bar{v}_1, \ldots, \bar{v}_N$ the sequence obtained from $v_1, \ldots, v_s$ by repeating v_i, n_i times. Let $0 = x_0 \leq x_1 \cdots \leq x_{r+1} = 1$, and $\bar{x}_1, \ldots, \bar{x}_N$ the sequence obtained from $x_1, \ldots, x_{r+1}$ by repeating x_i, m_i times.

D the set of all $\bar{x}_i$ such that:

$$\bar{v}_i < \bar{x}_i < \bar{v}_{i+n} \qquad i = 1, \ldots, N$$

where the right side is ignored if $i + m > N$.

We define $\Phi_m \begin{pmatrix} \bar{x}_1, \ldots, \bar{x}_N \\ \bar{v}_1, \ldots, \bar{v}_N \end{pmatrix}$ as does Michelli [4, eq (16)], and

note that for points in D, $\Phi_m \begin{pmatrix} \bar{x}_1, \ldots, \bar{x}_N \\ \bar{v}_1, \ldots, \bar{v}_N \end{pmatrix} > 0$. (see Schumaker [5], Theorem 4.78)

$$\text{Set} \quad u_{N+1}(x) = \sum_{i=0}^{s} (-1)^{g_i} \int_{v_i}^{v_{i+1}} \Phi_m(x, v)dv = G(x).$$

In D let $x^* = (x_1 \cdots x_r)$

$$F(x^*, x) = \frac{U\begin{pmatrix} u_1 & \cdots & u_N & u_{N+1} \\ \bar{x}_1 & \cdots & \bar{x}_N & x \end{pmatrix}}{U\begin{pmatrix} u_1 & \cdots & u_N \\ \bar{x}_1 & \cdots & \bar{x}_N \end{pmatrix}};$$

$$= G(x) + \sum_{i=1}^{N} b_i(x^*)\, u_i(x)$$

where

$$U\begin{pmatrix} u_1 & \cdots & u_N \\ \bar{x}_1 & \cdots & \bar{x}_N \end{pmatrix} = \Phi_m \begin{pmatrix} \bar{x}_1, \ldots, \bar{x}_N \\ \bar{v}_1, \ldots, \bar{v}_N \end{pmatrix} = \det\{u_i(\bar{x}_j);\ i, j = 1, \ldots, N\}$$

with the usual convention in case of coincidence among the $\bar{x}_i$'s (see Schumaker [5], §4.10)

$$Q(x^*) = \int_0^1 F(x^*, x)dx.$$

Then we prove the following:

Theorem 1. Assume m_i $i = 1, \ldots, r$ are even, and the set D is non-empty. Then $\min_{x^* \in D} Q(x^*)$, is attained for some $x^* \in D$.

Theorem 2. At the minimum of $Q(x^*)$, $x^* = (x_1, \ldots, x_r)$, we have

$$0 = x_0 < x_1 < x_2 < \cdots < x_r < x_{r+1} = 1.$$

Assuming Theorem 1 and Theorem 2 proved, we then prove:

Theorem 3. The equations (6) have a solution.

Thus the Main Theorem is established.

References

1. Barrar, R.B. and Loeb, H.L., Fundamental Theorem of Algebra for Monosplines and Related Results. Siam J. Numer. Anal. 17 (1980) pp. 874–882.

2. Barrar, R.B.; Bojanov, B.D.; and Loeb, H.L. Generalized Polynomials of Minimal Norm, to appear J. Approx. Theory.

3. Barrar, R.B. and Loeb, H.L. Oscillating Tchebycheff Systems, J. Approx. Theory 31 (1981) pp. 188–197.

4. Micchelli, C.A. The Fundamental Theorem of Algebra for Monosplines with Multiplicities, in Linear Operators and Approximation. P.L. Butzer et al., eds. Birkhaüser Verlag, Basel, Switzerland, 1972, pp. 419–430.

5. Schumaker, L.L. Spline Functions: Basic Theory. Wiley, New York, 1980.

6. Zhensykbaev, A.A. Fundamental Theorem of Algebra for Monosplines with Multiple Nodes, to appear J. Approx. Theory.

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