A Variant of the Krylov-Lanczos Method

for Bivariate Trigonometric Interpolation

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1 Introduction

Let a smooth bivariate function $x(s,t)$ be given on the unit square $U = [0,1]\times[0,1]$. We consider its sine double series expansion

$$\begin{aligned} x(s,t) &\sim 4\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \hat{x}_{kl}\sin k\pi s \sin l\pi t & (1)\\ \hat{x}_{kl} &= \int_0^1\int_0^1 x(\sigma,\tau)\sin k\pi\sigma \sin l\pi\tau d\tau d\sigma & (2)\end{aligned}$$

and its corresponding bivariate sine interpolant

$$\begin{aligned} P_{MN}x(s,t) &= 4\sum_{k=1}^{M}\sum_{l=1}^{N} x^*_{kl}\sin k\pi s \sin l\pi t & (3)\\ x^*_{kl} = x^*_{kl,MN} &= \frac{1}{(M+1)(N+1)}\sum_{i=1}^{M}\sum_{j=1}^{N} x(s_i,t_j)\sin k\pi s_i \sin l\pi t_j & (4)\end{aligned}$$

where $s_i = \frac{i}{M+1}$ $(i = 0,\ldots,M+1)$, $t_j = \frac{j}{N+1}$ $(j = 0,\ldots,N+1)$ are equidistant knot sequences. We have shown in [1,2,4] that the rate of convergence of the expansion (1) (measured in L_2 or L_∞-norm) depends only on the behaviour of the function $x(s,t)$ and of its derivatives along the boundary of U.

The following lemma gives the explicit relationship between smoothness and boundary behaviour of a function and the order of decay of its Fourier coefficients.

It can be proved either by representing the function as a convolution with polynomials whose Fourier coefficients are known [4] or directly by an iterated integration by parts argument, as in [3].

Lemma 1 *If* $x(s,t) \in C^{(2p+1,2p+1)}(U)$ *and*

$$x^{(2k,0)}(0,t) = x^{(2k,0)}(1,t) = 0 \qquad k = 0,\dots,p-1, \quad 0 \le t \le 1 \qquad (5)$$

$$x^{(0,2l)}(s,0) = x^{(0,2l)}(s,1) = 0 \qquad l = 0,\dots,p-1, \quad 0 \le s \le 1 \qquad (6)$$

then

$$\hat{x}_{kl} = \mathcal{O}\left(\frac{1}{(kl)^{2p+1}}\right) \quad (k,l \to \infty)$$

which ensures rapid convergence of the Fourier double series of $x(s,t)$. –

If $x \in C^{(2p+1,2p+1)}(U)$ but conditions (5), (6) are not satisfied then the *blended bivariate Lidstone interpolant*

$$\begin{aligned} y(s,t) &= \sum_{k=0}^{p-1} \left(x^{(2k,0)}(0,t)A_k(1-s) + x^{(2k,0)}(1,t)A_k(s)\right) \\ &+ \sum_{l=0}^{p-1} \left(x^{(0,2l)}(s,0)A_l(1-t) + x^{(0,2l)}(s,1)A_l(t)\right) \\ &- \sum_{k=0}^{p-1}\sum_{l=0}^{p-1} \Big(x^{(2k,2l)}(0,0)\cdot A_k(1-s)A_l(1-t) \\ &\qquad + x^{(2k,2l)}(0,1)\cdot A_k(1-s)A_l(t) \\ &\qquad + x^{(2k,2l)}(1,0)\cdot A_k(s)A_l(1-t) \\ &\qquad + x^{(2k,2l)}(1,1)\cdot A_k(s)A_l(t)\Big) \end{aligned}$$

provides a transfinite correction to $x(s,t)$, i. e. $\tilde{x}(s,t) = x(s,t) - y(s,t)$ satisfies conditions (5), (6) [1,2]. Here $A_k(s)$ denote the Lidstone polynomials which are recursively defined by

$$\begin{aligned} A_0(s) &= s \\ A''_{k+1}(s) &= A_k(s) \\ A_{k+1}(0) &= A_{k+1}(1) = 0 \quad (k \in \mathcal{N}). \end{aligned}$$

In [4] we have derived L_∞ error estimates for Fourier partial sum errors of smooth functions, whose corrections are elements of the *Korobov space*

$$E^{2p+1}(U) = \{x : U \to \mathcal{R} \;:\; \hat{x}_{kl} = \mathcal{O}(|kl|^{-2p-1}) \quad (k,l \to \infty)\}.$$

We have shown that in order to get good error estimates with few terms in the partial sum it is useful to include all terms with coefficients $\hat{x}_{kl}$ of magnitude up to a given bound. Because $\hat{x}_{kl} = \mathcal{O}(|kl|^{-2p-1})$ we obtain hyperbolic partial sums of the form $\sum\sum_{kl\le N}$ which yield almost the same error bounds as the tensor product sums $\sum_{k=1}^{N}\sum_{l=1}^{N}$ but appearently with fewer terms. We have derived our results in the setting of classical Fourier series with sine and cosine terms. Modifying the arguments in [4] slightly it is easy to prove the following estimates for pure sine series:

Theorem 1 *Let $x(s,t) \in E^{2p+1}(U)$. Then*

$$\left\| x(s,t) - 4\sum_{k=1}^{N}\sum_{l=1}^{N} \hat{x}_{kl} \sin k\pi s \sin l\pi t \right\|_\infty = \mathcal{O}(N^{-2p}) \quad (N\to\infty) \tag{7}$$

and

$$\left\| x(s,t) - 4\sum\sum_{kl\le N} \hat{x}_{kl} \sin k\pi s \sin l\pi t \right\|_\infty = \mathcal{O}\left(\frac{\ln N}{N^{2p}}\right) \quad (N\to\infty). \tag{8}$$

In this paper we will derive similar estimates for interpolatory approximants.

2 The Method of Aliasing for Bivariate Sine Interpolants

A standard technique for obtaining error estimates for trigonometric interpolants from Fourier partial sum errors is to express the coefficients of the trigonometric interpolant in terms of the Fourier coefficients of the considered function [11]. This technique is also applicable in the bivariate setting to be dealt with here. The so-called *aliasing method* is based on discrete orthogonality relations and on the condition that the Fourier series of a function is absolutely convergent to the function. The orthogonality relations for sine functions are as follows:

Lemma 2 *Let $l, \lambda, N \in \mathcal{N}$, $1 \le l \le N$. Let $s_k = \frac{k}{N+1}$ $(0 \le k \le N+1)$. Then*

$$\sum_{k=1}^{N} \sin l\pi s_k \sin \lambda\pi s_k = \begin{cases} \frac{1}{2}(N+1) & \text{if } \lambda = 2\nu(N+1)+l,\ \nu\in\mathcal{N}_0 \\ -\frac{1}{2}(N+1) & \text{if } \lambda = 2\nu(N+1)-l,\ \nu\in\mathcal{N} \\ 0 & \text{otherwise.} \end{cases}$$

Proof.

$$\begin{aligned} &\sum_{k=1}^{N} \sin l\pi s_k \sin \lambda\pi s_k \\ &= \sum_{k=1}^{N} \frac{1}{2i}\left(\exp\left(\frac{2\pi ikl}{2N+2}\right) - \exp\left(-\frac{2\pi ikl}{2N+2}\right)\right)\cdot \\ &\qquad \frac{1}{2i}\left(\exp\left(\frac{2\pi ik\lambda}{2N+2}\right) - \exp\left(-\frac{2\pi ik\lambda}{2N+2}\right)\right) \\ &= -\frac{1}{4}\sum_{k=0}^{2N+1} \exp\left(\frac{2\pi ik(l+\lambda)}{2N+2}\right) + \frac{1}{4}\sum_{k=0}^{2N+1} \exp\left(\frac{2\pi ik(l-\lambda)}{2N+2}\right). \end{aligned}$$

The lemma now follows from the discrete orthogonality relations for the n-th unit roots $\exp\left(\frac{2\pi i k}{n}\right)$. ∎

A sufficient condition for uniform convergence of the Fourier double series can be given as follows:

Lemma 3 *Let $x(s,t) \in E^{2p+1}(U)$ and $p \geq 1$. Then*

$$x(s,t) = 4\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \hat{x}_{kl} \sin k\pi s \sin l\pi t \qquad \text{for all } (s,t) \in U$$

with uniform convergence on U.

Proof. The terms of the series

$$S = \sum_k \sum_l \frac{C}{k^{2p+1} l^{2p+1}}$$

are upper bounds for the terms of

$$\sum_k \sum_l |\hat{x}_{kl} \sin k\pi s \sin l\pi t|$$

and S converges [4]. ∎

We can now express the discrete Fourier coefficients x^*_{kl} by $\hat{x}_{kl}$:

Theorem 2 *Let $x(s,t) \in E^{2p+1}(U), p \geq 1$ and let $x^*_{kl} = x^*_{kl,MN}$ be the coefficients of the $M \times N$ point sine interpolant for $x(s,t)$ as defined in (4). Then*

$$\begin{aligned} x^*_{kl} &= \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \hat{x}_{2\mu(M+1)+k,2\nu(N+1)+l} - \sum_{\mu=0}^{\infty}\sum_{\nu=1}^{\infty} \hat{x}_{2\mu(M+1)+k,2\nu(N+1)-l} \\ &- \sum_{\mu=1}^{\infty}\sum_{\nu=0}^{\infty} \hat{x}_{2\mu(M+1)-k,2\nu(N+1)+l} + \sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} \hat{x}_{2\mu(M+1)-k,2\nu(N+1)-l}. \end{aligned}$$

Proof. We observe that the Fourier double series is absolutely convergent for $x(s,t)$ and obtain by the discrete orthogonality relations:

$$\begin{aligned} x^*_{kl} &= \frac{1}{(M+1)(N+1)} \sum_{i=1}^{M}\sum_{j=1}^{N} x(s_i,t_j) \sin k\pi s_i \sin l\pi t_j \\ &= \frac{4}{(M+1)(N+1)} \sum_{i=1}^{M}\sum_{j=1}^{N} \left(\sum_{q=1}^{\infty}\sum_{r=1}^{\infty} \hat{x}_{qr} \sin q\pi s_i \sin r\pi t_j\right) \sin k\pi s_i \sin l\pi t_j \\ &= \frac{4}{(M+1)(N+1)} \sum_{q=1}^{\infty}\sum_{r=1}^{\infty} \hat{x}_{qr} \left(\sum_{i=1}^{M} \sin q\pi s_i \sin k\pi s_i\right)\left(\sum_{j=1}^{N} \sin r\pi t_j \sin l\pi t_j\right) \\ &= \sum_{\substack{q=1\\ q=2\mu(M+1)+k \\ r=2\nu(N+1)+l}}^{\infty}\sum_{r=1}^{\infty} \hat{x}_{qr} - \sum_{\substack{q=1\\ q=2\mu(M+1)-k \\ r=2\nu(N+1)+l}}^{\infty}\sum_{r=1}^{\infty} \hat{x}_{qr} - \sum_{\substack{q=1\\ q=2\mu(M+1)+k \\ r=2\nu(N+1)-l}}^{\infty}\sum_{r=1}^{\infty} \hat{x}_{qr} + \sum_{\substack{q=1\\ q=2\mu(M+1)-k \\ r=2\nu(N+1)-l}}^{\infty}\sum_{r=1}^{\infty} \hat{x}_{qr} \end{aligned}$$

by the discrete orthogonality relations. ∎

3 Error Estimates for Sine Interpolants

Error estimates for trigonometric interpolants are usually derived from the corresponding Fourier partial sum errors. This approach is also possible in the bivariate case. The details in our setting are as follows.

Lemma 4 *Let $x(s,t) \in E^{2p+1}(U), p \geq 1$. For $M, N \in \mathcal{N}$ let $x^*_{kl} = x^*_{kl,MN}$ be the coefficients of its sine interpolant as defined in (4) and $\hat{x}_{kl}$ be its Fourier coefficients as given in (2). Let $I \subseteq [1,M] \times [1,N]$ be an index set. Then*

$$\left\|\sum\sum_{(k,l)\in I} \hat{x}_{kl} \sin k\pi s \sin l\pi t - \sum\sum_{(k,l)\in I} x^*_{kl} \sin k\pi s \sin l\pi t\right\|_\infty \leq \sum\sum_{(k,l)\in \mathcal{N}\times\mathcal{N}-I} |\hat{x}_{kl}|.$$

Proof. By theorem 2 we have

$$\begin{aligned}
&\left\|\sum\sum_{(k,l)\in I} \hat{x}_{kl} \sin k\pi s \sin l\pi t - \sum\sum_{(k,l)\in I} x^*_{kl} \sin k\pi s \sin l\pi t\right\|_\infty \\
&= \Bigg\|\sum\sum_{(k,l)\in I}\Big(\hat{x}_{kl} - \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \hat{x}_{2\mu(M+1)+k,2\nu(N+1)+l} \\
&\qquad + \sum_{\mu=0}^{\infty}\sum_{\nu=1}^{\infty} \hat{x}_{2\mu(M+1)+k,2\nu(N+1)-l} \\
&\qquad + \sum_{\mu=1}^{\infty}\sum_{\nu=0}^{\infty} \hat{x}_{2\mu(M+1)-k,2\nu(N+1)+l} \\
&\qquad - \sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} \hat{x}_{2\mu(M+1)-k,2\nu(N+1)-l}\Big) \cdot \sin k\pi s \sin l\pi t\Bigg\|_\infty \\
&\leq \sum\sum_{(k,l)\in I} \Bigg(\sum_{\substack{\mu=0 \\ (\mu,\nu)\neq(0,0)}}^{\infty}\sum_{\nu=0}^{\infty} |\hat{x}_{2\mu(M+1)+k,2\nu(N+1)+l}| \\
&\qquad + \sum_{\mu=0}^{\infty}\sum_{\nu=1}^{\infty} |\hat{x}_{2\mu(M+1)+k,2\nu(N+1)-l}| \\
&\qquad + \sum_{\mu=1}^{\infty}\sum_{\nu=0}^{\infty} |\hat{x}_{2\mu(M+1)-k,2\nu(N+1)+l}| \\
&\qquad + \sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} |\hat{x}_{2\mu(M+1)-k,2\nu(N+1)-l}|\Bigg) \\
&\leq \sum\sum_{(k,l)\in \mathcal{N}\times\mathcal{N}-I} |\hat{x}_{kl}|.
\end{aligned}$$

Note for the last appraisal that none of the terms has an index in the range $[1,M] \times [1,N]$ since $\mu \neq 0$ or $\nu \neq 0$ in all sums. ■

The main result now follows immediately from all prerequisites developed:

Theorem 3 *Let $x(s,t) \in E^{2p+1}(u), p \geq 1$ and $P_{NN}x(s,t)$ be the sine interpolant with N^2 knots for $x(s,t)$, as defined in (3). Then*

$$\|x(s,t) - P_{NN}x(s,t)\|_\infty = \mathcal{O}\left(\frac{1}{N^{2p}}\right).$$

Proof. By theorem 1 and lemma 4 we get

$$\begin{aligned}
&\|x(s,t) - P_{NN}x(s,t)\|_\infty \\
\leq\ & \left\| x(s,t) - 4\sum_{k=1}^{N}\sum_{l=1}^{N} \hat{x}_{kl} \sin k\pi s \sin l\pi t \right\|_\infty \\
+\ & \left\| 4\sum_{k=1}^{N}\sum_{l=1}^{N} \hat{x}_{kl} \sin k\pi s \sin l\pi t - 4\sum_{k=1}^{N}\sum_{l=1}^{N} x^*_{kl} \sin k\pi s \sin l\pi t \right\|_\infty \\
\leq\ & 8 \sum_{\substack{k=1\\(k,l)\notin[1,N]\times[1,N]}}^{\infty}\sum_{l=1}^{\infty} |\hat{x}_{kl}| \\
\leq\ & \sum_{\substack{k=1\\(k,l)\notin[1,N]\times[1,N]}}^{\infty}\sum_{l=1}^{\infty} \frac{8}{(kl)^{2p+1}} \\
=\ & \mathcal{O}\left(\frac{1}{N^{2p}}\right). \qquad \blacksquare
\end{aligned}$$

Instead of evaluating the complete tensor product interpolant it is sufficient for good error estimates to compute the hyperbolic sum $\sum_{kl \leq N}$, which depends on less coefficients, but yields an analogue error to (8). The resulting approximant will not be interpolating in general, though.

Corollary 1

$$\left\| x(s,t) - 4\sum_{kl \leq N} x^*_{kl} \sin k\pi s \sin l\pi t \right\|_\infty = \mathcal{O}\left(\frac{\ln N}{N^{2p}}\right) \quad (N \to \infty).- \tag{9}$$

The proof is carried out exactly as for theorem 3.

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