CONSTRUCTIVE THEORY OF FUNCTIONS'87

SOFIA, 1988

ON HÖLDER CONTINUITY OF THE RIEMANN MAPPING FUNCTION

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I. Introduction. Let G be an arbitrary simply connected region in the extended complex plane $\mathbb{C}_z^*$ and $\bar{G}$ its closure, $\Omega = \mathbb{C}_z^* \setminus \bar{G}$, $w = \varphi(z)$ is the Riemann function which gives a conformal and univalent mapping of G onto the unit disk $D_w = \{|w| < 1\}$. For cases when the function $\varphi(z)$ has a continuous extention to the homeomorphism between $\bar{G}$ and a closure $\bar{D}_w$ and generates the homeomorphism $\partial G \leftrightarrow \partial D_w$, many authors have investigated the following problem A: what are geometrical conditions for $L = \partial G$, ensuring the Hölder continuity of $\varphi(z)$ (also for Hölder continuity of the inverse function $z = \psi(w)$); what are the best possible degrees of a Hölder continuity for a given L ? Obtaining the solution of the problem A is essential for the approximation theory in the complex domain, for numerical methods of the complex analysis and various applications of conformal mappings.

Note that we can say only about a homeomorphism between the compactification G of a domain G by Caratheodory prime ends and $\bar{D}_w$ in a general case of an arbitrary simply connected domain. Therefore , such a putting of the problem assumes cosidering mainly the domains whose boundaries contain prime ends of the first type only (for example, Jordan domains).

If G is a bounded Jordan domain in $\mathbb{C}_z^*$ and $w = \Phi(z)$ is a Riemann function that maps conformally and univalently Ω onto $\Omega' = \mathbb{C}_w^* \setminus \bar{D}_w$; $\Psi(w) = \Phi^{-1}(w)$, then the problem A is closely connected with the following problem B (see [2, problem 6.60]): let $\psi(w)$ has some continuous properties on ∂D_w , what can be said about continuous properties of $\Psi(w)$ on ∂D_w and connections between continuous properties of $\varphi(z)$ and $\Phi(z)$ on ∂G ?

The purpose of the paper is to show that solutions of the problems A and B obtained by some authors, could be derived from general relations of the local thory of a distance distortion under conformal mappings onto the exterior or interior of the unit disk (see [3], [4]) by the direct estimation of the reduced module of the separating curves family.

2. Definitions and notations. Let $G \subset \mathbb{C}^*$ be an arbitrary simply connected domain, let $L = \partial G \not\ni \infty$, and let $\tilde{G}$ its compactification by Caratheodory prime ends. We denote the elements of $\tilde{G}$ by capital letters $Z, \mathfrak{z}$, etc.; the body of the prime end $Z \in \tilde{G}$ - by $|Z|$. Let us write $z \in Z$, if $z \in |Z|$. For the interior point $z \in G$ we always have $z = |Z|$. We need the notion of the module of the separating curves family in G reduced relative to a fixed point $a \in G$ (see [3], [4]). It was defined by the following relations

$$\mu_a(Z,\mathfrak{z},G) = \lim_{r\to 0}\left[m(Z,\mathfrak{z},D_r) + \frac{I}{2\pi}\ln r\right] , \quad (a \neq \infty), \qquad (I)$$

$$\mu_\infty(Z,\mathfrak{z},G) = \lim_{r\to \infty}\left[m(Z,\mathfrak{z}, D_r) - \frac{I}{2\pi}\ln r\right] , \qquad (2)$$

where $D_r = \{|z-a| \leq r$ as $a \neq \infty$, $D_r = \{|z| \geq r\}$ as $a = \infty$, $m(Z,\mathfrak{z}, D_r)$ is the module of a family of curves separating prime ends Z and $\mathfrak{z}$ from D_r in G.

Some other notations:

- R : interior conformal radius of a domain G relative to $a \in G$;
- $\mathbb{C}^*$: extended complex plane;
- c : conformal capacity of the complement $\Omega = \mathbb{C}^* \setminus \bar{G}$;
- $\asymp$: weak equivalence sign;
- $\preccurlyeq$: orderial inequality sign.

3. Criteria of Hölder continuity. The properties of the Riemann mapping function are completely determined by its behaviour near the boundary of the domain under cosideration so the study of mappings of the unit disk D_w onto the bounded simply connected domain G and the study of a mapping $\Omega' = \mathbb{C}^*_w \setminus \bar{D}_w$ onto the simply connected domain $\Omega \ni \infty$ are problems of the same type. But we will consider them separately for the convenience of using and taking the different normalization into account.

Consider the following mappings:

(I) $w = \varphi(z)$ is the Riemann function which maps an arbitrary simply connected domain $\Omega \subset \mathbb{C}^*_z$, $\infty \in \Omega$ onto $\Omega' = \mathbb{C}^*_w \setminus \bar{D}_w$ and is normalized by $\varphi(\infty) = \infty$; $\lim_{z\to\infty}\varphi(z)/z = c^{-I} > 0$ ($c = \mathrm{cap}(\mathbb{C}^*_z \setminus \Omega)$). Let also $z = \Psi(w) = \varphi^{-I}(w)$.

(2) $w = \varphi(z)$ is the Riemann function which maps an arbitrary simply connected domain $G \subset \mathbb{C}^*_z$ onto D_w and is normalized by $\varphi(a) = 0$, $\varphi'(a) = I/R > 0$ (R is the conformal radius G relative to $a \in G$). Let also $z = \psi(w) = \varphi^{-1}(w)$.

Theorem I. The following two statements are equivalent:

(i) $\Psi(w)$ can be continuously extended to $\partial\Omega'$ and belongs to the class Lip α , $\alpha \in (0,I]$ in Ω' ;

(ii) for any boundary prime ends $Z \in \tilde{\Omega}$, $\mathfrak{Z} \in \tilde{\Omega}$ and any points $z \in |Z|$ and $\zeta \in |\mathfrak{Z}|$ the following inequality is true

$$\mu_\infty(Z,\mathfrak{Z},\Omega) \leq \frac{I}{\pi\alpha} \ln \frac{I}{|z-\zeta|} + M_I , \qquad (3)$$

where constant M_I can depend on Ω and α only.

Remark I. It is clear that the continuous extention of $\Psi(w)$ onto $\partial\Omega'$ is possible only for the case when Ω has prime ends of the first type. Therefore, we can consider $z = |Z|$ and $\zeta = |\mathfrak{Z}|$ in (3).

Theorem I can be easy derived from the more general result contained in [3] : if $w, \tau \in \partial\Omega'$, $w \xrightarrow{\Psi} Z$, $\tau \xrightarrow{\Psi} \mathfrak{Z}$, $\mu_\infty = \mu_\infty(Z,\mathfrak{Z},\Omega)$ then

$$|w - \tau| \asymp \exp(-\pi\mu_\infty) , \qquad (4)$$

where constants in the relation " $\asymp$ " are termed through c. To get more accurate estimation of the Hölder constant, the precise equality (see [4] or [5]) can be used:

$$|w - \tau| = \frac{4}{c}\sqrt{c - \exp(-2\pi\mu_\infty)}\ \exp(-\pi\mu_\infty) \qquad (5)$$

under $|w - \tau| \leq I,3$. If we replace $\mu_\infty = \mu_\infty(Z,\mathfrak{Z},\Omega)$ in (5) by the reduced module $\mu_\infty(\overline{Z\mathfrak{Z}},\Omega)$ of the family of curves separating the part of the boundary $\partial\Omega$ containing all the prime ends between Z and $\mathfrak{Z}$, then the relation (5) will be valid for all w and τ [5] .

Proof of theorem I. Really, (i) $\rightarrow$ (ii). Taking into consideration remark I and using (4) we have

$$|z - \zeta| = |\Psi(w) - \Psi(\tau)| \lesssim |w - \tau|^\alpha \lesssim e^{-\pi\alpha\mu_\infty} , \qquad (6)$$

and

$$\mu_\infty(Z,\mathfrak{Z},\Omega) \leq \frac{I}{\pi\alpha} \ln \frac{I}{|z - \zeta|} + M_I .$$

From the other side , (ii) $\rightarrow$ (i). As it is known (see, for example [9]), it is enough to verify the Hölder condition only on the boundary $\partial\Omega'$. From (3) and (4), again we have under all $w,\tau\in\partial\Omega'$:

$$\left|\Psi(w)-\Psi(\tau)\right| = |z-\zeta| \leq e^{\pi\alpha M_I}\, e^{-\pi\alpha\mu_\infty}\lesssim|w-\tau|^{\alpha}, \qquad (7)$$

and that is the end of the proof.

Theorem 2. The following two statements are equivalent:

(i) $w=\varphi(z)$ can be continuously extended to $\partial\Omega$ and belongs to the class Lip α, $\alpha\in[0,I]$ in $\bar{\Omega}$;

(ii) for any points z and $\zeta\in\partial\Omega$ and for any prime ends Z and $\mathcal{Z}$ such that $z\in|Z|$ and $\zeta\in|\mathcal{Z}|$ the following inequality is true

$$\mu_\infty(Z,\mathcal{Z},\Omega) \geqslant \frac{\alpha}{\pi}\ln\frac{I}{|z-\zeta|} + M_2 \qquad (8)$$

where constant M_2 can depend on Ω and α only.

The proof is completely analogous to the one for the theoremI. Notice that for the Jordan boundary $\partial\Omega$ we always have

$$\mu_\infty(Z,\mathcal{Z},\Omega) \geqslant \frac{I}{2\pi}\ln\frac{I}{|z-\zeta|} + M_2 ,$$

so $\varphi\in$ LipI/2 and it is natural to consider only $\alpha\in[I/2,I]$.

Considering the mappings $\varphi(z)$ and $\psi(w)$ it should be taken into account that in this case relations (4) and (5) have a form

$$|w-\tau| \asymp \exp(-\pi\mu_a) ; \qquad (4')$$

$$|w-\tau| = 4R\sqrt{R^{-I}-e^{-2\pi\mu_a}}\ \exp(-\pi\mu_a) , \quad (|w-\tau|<I,3) \quad (5')$$

where $w,\tau\in\partial D_w$, $w\xrightarrow{\Psi}Z$; $\tau\xrightarrow{\Psi}\mathcal{Z}$, $\mu_a=\mu_a(Z,\mathcal{Z},G)$ and constants in the relation "$\asymp$" are termed through R, and the equality (5') will be true for all w,τ after changing $\mu_a(Z,\mathcal{Z},G)$ by $\mu_a(\overline{Z\mathcal{Z}},G)$ (see [5]).

Theorem I'. The following two statement are equivalent:

(i) $z=\psi(w)$ can be continuously extended to $\{|w|=I\}$ and belongs to the class Lip α, $\alpha\in[0,I]$ in $\bar{D}_w$;

(ii) for any boundary prime ends $Z\in\tilde{G}$ and $\mathcal{Z}\in\tilde{G}$ and any two points $z\in|Z|$ and $\zeta\in|\mathcal{Z}|$ the following inequality is true

$$\mu_a(Z,\mathcal{Z},G) \leq \frac{I}{\pi\alpha}\ln\frac{I}{|z-\zeta|} + M_I , \qquad (3')$$

where the constant M_I can depend on G and α only.

Theorem 2′. The following two statements are equivalent:

(i) $w = \varphi(z)$ can be continuously extended to ∂G and belongs to the class Lip α, $\alpha \in [0,I]$ in $\overline{G}$;

(ii) for any points z and $\zeta \in \partial G$ and any prime ends Z and $\mathcal{Z}$ such that $z \in |Z|$ and $\zeta \in |\mathcal{Z}|$ the following inequality is true

$$\mu_a(Z,\mathcal{Z},G) \geqslant \frac{\alpha}{\pi} \ln \frac{I}{|z-\zeta|} + M_2 \qquad (8')$$

where the constant M_2 can depend on G and α only.

Remark 2. Reduced moduli $\mu_\infty(Z,\mathcal{Z},\Omega)$ and $\mu_a(Z,\mathcal{Z},G)$ in the relations (3),(3′),(8),(8′) can be replaced by $\mu_\infty(\overline{Z\mathcal{Z}},\Omega)$ and $\mu_a(\overline{Z\mathcal{Z}},G)$ accordingly.

Remark 3. The constants in statements (i) and (ii) of theorems I and I′, 2 and 2′ are interconnected and can be easy evalueted one through another and constants in relations (4)-(5) and (4′)-(5′).

4. The α-wedge condition. Following Lesley [7] we will say that the boundary $L = \partial G$ satisfies the interior α-wedge condition if there exist $r > 0$ and $\alpha \in [0,I]$ such that for each point $z \in L$ the radius r with the corner $\alpha\pi$ and a vertex at z belongs to $\widetilde{G}$. We will say that L satisfies the exterior α-wedge condition, if for each $z \in L$ there exist such a sector belonging to $C^* \setminus G$. In the case when G is an interior of a closed Jordan curve L, Ω is its exterior, φ, ψ, Φ, Ψ, are the same mappings as before, Lesley [7] obtained the following sufficient conditions of the Hölder continuity (We have formulated them for the arbitrary simply connected domain):

Theorem 3 ([7]). Let L satisfies the interior α-wedge condition. Then $\psi \in$ Lipα .

Theorem 4 ([7]). Let L satisfies the exterior α-wedge condition. Then $\varphi \in$ Lip I/2-α .

Both results (and their full analogs for mappings Ψ and Φ) can be easily derived from the theorems I′ and 2′ by the direct estimation of reduced moduli μ_a and μ_∞ . To get the upper estimate for μ_a the preliminary upper estimate of the module $m(z,\zeta,D_r)$ must be made. We make it by the proper choice of the metric. After the tending to limit under $r \to 0$ we have inequality (3′) which leads to the statement $\psi \in$ Lipα (by theorem 3′).

To prove theorem 4 the estimates of μ_a from the below are needed. It is enough to prolong the separating curves in such a way that the resulting family separates points z and ζ from the point a in the exterior of the α-wedge with the vertex at z. The module of the family decreases after such an operation and can be easy estimated. The strict degree and the coefficient of the Hölder continuity are clearly defined through the interior conformal radius of the domain. For the mappings Ψ and Φ the same outline of the proof is good.

Lesley's proof is much more complicated.

5. Problem B. Theorems I - 2 and I′- 2′ permit to prove in the same simple manner the following result by Lesley [8] that solves a problem, putting in [I, Problem 6.60].

Theorem 5. (Lesley [8]) If $\psi \in \mathrm{Lip}\,\alpha$ on $\overline{D}_w$, $\alpha \in [0,I]$ then $\Phi \in \mathrm{Lip}\ I/2\text{-}\alpha$ in the kernel of Ω (Details about the definition of the Kernel see at [8]).

Theorem 5 is also the corollary of theorems I - I′, 2 - 2′. The outline of the proof: $\psi \in \mathrm{Lip}\,\alpha \Rightarrow (3') \Rightarrow$ the prolongation of the curves family, defining $\mu_a(\overline{z\zeta}, G)$ to the curves family, separating the part of the boundary $\overline{z\zeta}$ from $\infty \Rightarrow (8) \Rightarrow$ theorem $2 \Rightarrow \Phi \in \mathrm{Lip}\frac{I}{2-\alpha}$.

The roles of the interior G of L and the exterior Ω can be changed into the exterior and interior correspondently. It leads to obvious nonessential alterations of the proof.

In conclusion we are going to draw attention to the possibility of obtaining in the same direct way some other results describing the boundary behaviour of conformal mappings, for example results by D.Gaier [6] .

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