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A MEAN VALUE THEOREM IN METRIC SPACES

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The well-known Mean Value Theorem says that if X and Y are Banach spaces, $U \subset X$ is a convex open set, $f : U \to Y$ is a differentiable function and for every point x in U $\|Df(x)\| \leq K$ holds, then f is a Lipschitzean mapping on U with constant K. In [1] Henry Cartan gave the following generalization of this theorem: If f is a function between the Banach spaces X and Y and f is differentiable on the connected open set U, moreover $\|Df(x)\| \leq K$ if $x \in U$, then $\|f(x)-f(u)\| \leq K\,\ell(\Gamma)$, where $x,u \in U$, Γ is a curve lying in U and having x and u as end points and $\ell(\Gamma)$ is the length of the curve Γ. In this paper we give an analogous result when X and Y are metric spaces. This result contains Cartan's Mean Value Theorem as a special case.

<u>Definitions 1</u>. Let (X,d) be a metric space, $g : [0,1] \to X$ a continuous function and $\Gamma := g([0,1])$. The curve Γ is said to be <u>rectifiable</u> if there is a positive number L such that the following inequality holds:

$$\sum_{i=1}^{n-1} d(g(t_{i+1}),\, g(t_i)) \leq L,$$

whenever $0 = t_1 < t_2 < \dots < t_n = 1$. Denote $\ell(\Gamma)$ the infimum of such numbers L. $\ell(\Gamma)$ is called the <u>length of Γ</u>. Let (X,d_1) and (Y,d_2) be metric spaces. The function $f : X \to Y$ is called <u>locally star-like Lipschitzean</u> at the point x with constant K if there is a neighbourhood U of x such that $d_2(f(x),\, f(u)) \leq K d_1(x,u)$, whenever $u \in U$.

Remark 1. If X and Y are Banach spaces then the continuous linear mapping $A : X \to Y$ is the Fréchet-derivative of f at the point x if and only if for every positive number ε the function $f-A$ is locally star-like Lipschitzean with constant ε.

Theorem. Let (X,d_1) and (Y,d_2) be metric spaces, $f : X \to Y$ a function and $g : [0,1] \to X$ a continuous function. Suppose that the curve $\Gamma := g([0,1])$ is rectifiable. If for each point $x \in \Gamma$ there exists a locally star-like Lipschitzean mapping $A_x : X \to Y$ with constant K /not depending on x/ such that for every positive number ε there is a neighbourhood U of x such that $d_2(f(x), f(u)) \leq d_2(A_x(x), A_x(u)) + \varepsilon d_1(x,u)$, whenever $u \in U$, then $d_2(f(a), f(b)) \leq K\,\ell(\Gamma)$, where $a := g(0)$ and $b := g(1)$.

Remark 2. When X and Y are Banach spaces, f differentiable at x and $A := Df(x)$ then we have that $\|f(x)-f(u)\| \leq$ $\leq \|A(x)-A(u)\| + \varepsilon\|x-u\|$, whenever $\|f(x)-f(u)-A(x-u)\| \leq \varepsilon\|x-u\|$, using the triangle inequality.

Proof. Let ε be an arbitrary positive number. For each $x \in \Gamma$ there is an U_x open neighbourhood of x such that $d_2(A_x(x),A_x(u)) \leq$ $\leq K\, d_1(x,u)$ and $d_2(f(x),f(u)) \leq d_2(A_x(x),A_x(u)) + \varepsilon d_1(x,u)$, whenever $u \in U_x$. So $d_2(f(x),f(u)) \leq (K+\varepsilon)\,d_1(x,u)$. /1/

As g is continuous the set $g^{-1}(U_x)$ contains an interval I_t for each point $t \in g^{-1}(x)$ wich is on open neighbourhood of t if $t \neq 0$ and $t \neq 1$. We show that there are finite intervals $I_{t_1},\dots,I_{t_n}$ with the following properties: $0 = t_1 < t_2 < \dots < t_n = 1$,

$$[0,1] \subset \bigcup_{i=1}^{n} I_{t_i} \quad \text{and} \quad I_{t_i} \cap I_{t_{i+1}} \neq \emptyset, \quad i=1,\dots,n-1.$$

Indeed, if $c := \sup\{k \in R \mid \exists\, t_1,\dots,t_n : 0 = t_1 < t_2 < \dots < t_n,$ $[0,k] \subset \bigcup_{i=1}^{m} I_{t_i},\ I_{t_i} \cap I_{t_{i+1}} \neq \emptyset,\ i=1,\dots,m-1\} < 1$ then

$[0,v[\ \subset \bigcup_{i=1}^{m} I_{t_i} \cup I_c$, where $]u,v[\ = I_c$ and $[0,u] \subset \bigcup_{i=1}^{m} I_{t_i}$,

which is a contradiction. Hence $c=1$ and $[0,1] \subset \bigcup_{i=1}^{k} I_{t_i} \cup I_1$

where $I_1 =]v,1]$ and $[0,v + \frac{1-v}{2}] \subset \bigcup_{i=1}^{k} I_{t_i}$.

Let $0 = t_1 < t_2 < \ldots < t_n = 1$, $[0,1] \subset \bigcup_{i=1}^{n} I_{t_i}$, $s_i \in I_{t_i} \cap I_{t_{i+1}}$, $t_i \leqslant s_i \leqslant t_{i+1}$ and $u_i := g(s_i)$ $(i=1,\ldots,n-1)$. Then

$$d_2(f(a),f(b)) \leqslant \sum_{i=1}^{n-1} [d_2(f(x_i),f(u_i))+d_2(f(u_i),f(x_{i+1}))] \leqslant$$

$$\leqslant (K+\varepsilon) \sum_{i=1}^{n-1} [d_1(x_i,u_i)+d_1(u_i,x_{i+1})] \leqslant (K+\varepsilon)\ \ell(\Gamma),$$

as $u_i \in U_{x_i} \cap U_{x_{i+1}}$ /i=1,...,n-1/ and the property /1/ holds.

Since ε is an arbitrary positive number, it follows that $d_2(f(a),f(b)) \leqslant K\ \ell(\Gamma)$. □

Let X be a metrizable topological linear space with a p-homogeneous norm /i.e. $p > 0$ and $\|tx\| = |t|^p \|x\|$, whenever $x \in X$ and t is a scalar/. Denote $S(x;r) := \{u \in X \mid \|x-u\| < r\}$ and $L(u;v) := \{(1-t)u+tv \mid t \in [0,1]\}$ $(u,v \in X)$. If $\{u,v\} \subset S(x;r)$ then $L(u,v) \subset S(x;2r)$. Indeed, $\|(1-t)u+tv\| \leqslant |1-t|^p\|u\|+|t|^p\|v\| \leqslant 2r$. Since every linear operator $A : X \to Y$ is continuous if and only if it is Lipschitzean /see [2]/ we could apply the Theorem given above when $f \in X \to X$ is continuously differentiable on an open set. But the fact is that the line segment $L(u,v)$ is not rectifiable if $p < 1$. For instance $L(0,v) = \{tv \mid t \in [0,1]\}$ and

$$(L(0,v)) \geqslant \sum_{i=1}^{n} \|\tfrac{1}{n}v\| = n^{1-p}\|v\| \to \infty \quad \text{if } n \to \infty.$$

So the following question arises:

Problem. In which metrizable topological linear spaces are there rectifiable curves at all?

References

1. Cartan, Henry: Calcul différentiel, formes différentielles. Hermann, Paris, 1967.

2. Rolewicz, Stefan: Metric linear spaces. PWN-POLISH SCIENTIFIC PUBLISHERS, WARSZAVA, 1972.

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