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ERROR ESTIMATE FOR BOX SPLINE INTERPOLATION

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Let V be a set of n vectors $v_j \in Z^d \setminus (0,..,0)$. The elements of V are not necesarily distinct but we assume that span $V = R^d$. The box spline M_V is defined as a distribution in R^d given by the rule $M_V : \phi \rightarrow \int_{[-\frac{1}{2},\frac{1}{2}]^d} \phi(\sum_{v_j \in V} \mu(j)v_j)\, d\mu$,

where $\mu = (\mu(1),..,\mu(n))$. An equivalent definition of M_V can be given via its Fourier transform $\widehat{M}_V(x) := \prod_{v \in V} \operatorname{sinc}(x.v)$,

where $x.v$ is the scalar product and $\operatorname{sinc}(2t) := \sin t \quad t$.

The box splines were introduced by de Boor and DeVore [1] and their basic properties have been described in [2] . For a survey of the recent stage of the box spline theory we refer to [3,4,5] .

Multivariate cardinal splines are defined as linear combinations of translates of a given box spline: $S_V := \operatorname{span}\left\{ M_V(.-j) \ ; \ j \in Z^d \right\}$. Here we investigate the space $S_{V,h} := \left\{ s(\frac{\cdot}{h}) \ ; \ s \in S_V \right\}$ of multivariate splines with meshsize h.

Let $f : R^d \rightarrow R$ be a function of power growth, i.e. there exists a constant $N=N(f)$ such that $f(x)=O(|x|^N)$, $|x| \rightarrow \infty$. We introduce the following i n t e r p o l a t i o n p r o b l e m (IP) :

for a given $\beta \in [-\frac{1}{2},\frac{1}{2})^d$ find a spline $I_h f \in S_{V,h}$ which is a

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function of power growth and satisfies the equations

(1) $$I_h f((j+\beta)h) = f((j+\beta)h), \quad \text{all } j \in Z^d.$$

A necessary condition for corectness of IP is the so - called "determinant condition" :

for every $W \subset V$ which forms a basis in R^d we have $|\det W| = 1$.

Let $P_{V,\beta}(z) := \sum_{j \in Z^d} M_V(\beta+j)\, e^{ij \cdot z}$ be the characteristic polynomial of IP for a given V and β. The following theorem takes place

Theorem 1. [5,6] Let V satisfy the determinant condition. Then IP is corect iff the polynomial $P_{V,\beta}(z)$ does not vanish on R^d.

From now on we assume that V and β satisfy the conditions of Theorem 1. The aim of this paper is to estimate the error $f - I_h f$ in L_p-norm. To obtain estimates without other assumptions on f except local summability and power growth, we use the average moduli of smoothness (also called τ-moduli)

$$\tau_k(f;G)_p := \| \omega_k(f;\cdot+G) \|_p ,$$

where $\omega_k(f;U) := \sup \{ |\Delta_h^k f(x)| \; ; \; [x,x+kh] \subset U \}$ is the usual modulus of smoothness for f on a given set $U \subset R^d$ and $[x,x+kh] := \{ y = x+tkh \; ; \; 0 \le t \le 1 \}$.

In the univariate case these moduli were introduced first by Sendov and Korovkin (see [13] for references). The multivariate moduli were introduced by Popov, Hristov and Ivanov [9,10,11,12], when G is a ball. It is easy to see that the basic properties of these moduli hold true also when G is a center symmetric convex body. In partiqular, the following inequality holds (see [11,12])

(2) $$\tau_k(f;G)_p \le c \left\{ \sigma^k \sum_{|\mu|=k} \| D^\mu f \|_p + \sum_{k<|\mu|\le d} \sigma^{|\mu|} \| D^\mu f \|_p \right\},$$

where $\sigma = \operatorname{diam} G$ and the constant c depends only on k and p; the second sum is vacant when $k \ge d$.

The main result of the paper is

Theorem 2. Let $k := \min \{ \#W \; ; \; \operatorname{span}(V \setminus W) \ne R^d \}$ and $G := [-\frac{1}{2}, \frac{1}{2}]^d + \operatorname{supp} M_V$. Then

$$\| f - I_h f \|_p \leq C \, \tau_k(f ; hG)_p$$

with a constant C depending only on $ß$ and V.

Using (2) we derive the following corollaries

Corollary 1. Let $d \leq k$ and $f \in W_p^k$. Then

$$\| f - I_h f \|_p = O(h^k) .$$

Corollary 2. Let $d > k$ and $f \in W_p^d$. Then

$$\| f - I_h f \|_p = O(h^k) .$$

1° The operator BS_V

We define the operator

(3) $$BS_V(f;x) := \sum_j M_V(j-ß) \, f(x+jh) - \sum_j M_V(j-\tfrac{x}{h}) \, f((j+ß)h) ,$$

where both sums are taken over all $j \in Z^d$ for which box spline M_V does not vanish. Obviously, this operator is linear. In this section we prove that $\operatorname{Ker} BS_V = S_{V,h}$.

Lemma 1. For every $f \in S_{V,h}$ we have

$$BS_V(f;x) = 0 , \quad \text{all } x \in R^d .$$

Proof. Let $f(x) = \sum_l a_l \, M_V(l-\tfrac{x}{h})$. Then

$$M_V(j-ß) \, f(x+jh) = \sum_l a_l \, M_V(l-j-\tfrac{x}{h}) \, M_V(j-ß).$$

Substitute $m=l-j$ we receive

$$M_V(j-\tfrac{x}{h}) \, f((j+ß)h) = \sum_l a_l \, M_V(l-j-ß) \, M_V(j-\tfrac{x}{h}) = \sum_l a_l \, M_V(m-ß) \, M_V(m-j-\tfrac{x}{h}).$$

Hence the subtraction of the sums in (3) is zero. □

Using linearity of BS_V, Lemma 1, (3) and (1) we obtain

$$BS_V(f;x) = BS_V(f-I_hf;x) = \sum_j M_V(j-ß) \, (f(x+jh) - I_hf(x+jh)).$$

Let $x=lh+u$, where $l \in Z^d$ and $u \in [-\tfrac{h}{2},\tfrac{h}{2})^d$. Then noting $g_j(u) := f(jh+u) - I_hf(jh+u)$ we receive

(4) $$\sum_j M_V(j-ß) \, g_{j+l}(u) = BS_V(f;lh+u) .$$

Let u be fixed and $g_j(u)$ $(j \in Z^d)$ be unknown quantities. Then from (4) we obtain the double infinite system

(5) $$\sum_j M_V(j-l-ß) \, g_j(u) = BS_V(f;lh+u) , \quad l \in Z^d .$$

The matrix of this system $A := (M_V(j-l-ß))_{j,l \in Z^d}$ is a Toeplitz matrix. It is the same as in the IP (see [5,6]). Under the assumption in the begining the characteristic polynomial $P_{V,\beta}(z)$ does not vanish and therefore there exists the inverse of A : $A^{-1} = (z_{j-l})_{j,l\in Z^d}$, where z_j $(j\in Z^d)$ are given by

$$z_j := \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{e^{-ij.x}}{P_{V,\beta}(x)}\, dx .$$

It follows from the last equality that z_j desay exponentialy as $|j|\to\infty$ and, in partiqular, there exists a constant C_1 depending only on V and ß such that

(6) $$\sum_{j\in Z^d} |z_j| \le C_1 .$$

The solution of (5) is $g_1(u) = \sum_j z_j\, BS_V(f;(l+j)h+u)$ and therefore substitute $x=lh+u$ we have

(7) $$f(x)-I_h f(x) = \sum_j z_j\, BS_V(f;x+jh) .$$

Finaly we prove $\mathrm{Ker}\, BS_V = S_{V,h}$. Immediately from Lemma 1 it follows that $S_{V,h}\subset \mathrm{Ker}\, BS_V$. Let $f\in \mathrm{Ker}\, BS_V$. Then from (7) we obtain $f(x) = I_h f(x)$ for all $x\in R^d$ and from the definition of $I_h f$ we have $f\in S_{V,h}$.

2° An estimate for BS_V

We use the following multivariate analog of Whitney theorem for polynomials (see [7,8])

Theorem 3. Let $G\subset R^d$ be a convex body and f be a bounded and measurable function on G. Then there exists a polynomial p of total degree k-1 such that for all $x\in G$

$$|f(x)-p(x)| = C_2\, \omega_k(f;G)$$

with a constant C_2 depending only on k and d.

It follows immediately from definition (3) that $BS_V(f;x)$ is a linear combination of values of f on the set $x+h(\beta+\mathrm{supp}\, M_V) \subset x+hG$. Hence using triangular inequality, $M_V \ge 0$ and the property $\sum_j M_V(.+j) = 1$ (see [2]) we obtain

$$
(8)\quad \begin{aligned} |BS_V(f;x)| &\le \sum_j M_V(j-\beta)\,|f(x+jh)| + \sum_j M_V(j-\tfrac{x}{h})\,|f((j+\beta)h)| \\ &\le \Big(\sum_j M_V(j-\beta) + \sum_j M_V(j-\tfrac{x}{h})\Big) \sup_{y\in(x+hG)} |f(y)| = 2 \sup_{y\in(x+hG)} |f(y)| . \end{aligned}
$$

Let k be from Theorem 2 and p be the polynomial from Theorem 3 for f on the set $(x+hG)$. Then

$$
(9)\qquad \sup_{y\in(x+hG)} |f(y)-p(y)| \le C_2\, \omega_k(f;x+hG) .
$$

From the other hand $S_{V,h}$ contain all polynomials of total degree $k-1$ (see [2]). Using Lemma 1, $p\in S_{V,h}$, (8) for $f-p$ and (9) we obtain

$$
(10)\qquad |BS_V(f;x)| = |BS_V(f-p;x)| \le 2C_2\, \omega_k(f;x+hG) .
$$

3° Proof of Theorem 2

From (7) and (10) we receive

$$
|f(x) - I_h f(x)| \le \sum_j |z_j|\; |BS_V(f;x+jh)| \le 2C_2 \sum_j |z_j|\, \omega_k(f;x+jh+hG).
$$

Hence using triangular inequality we prove the theorem

$$
\begin{aligned} \| f-I_h f \|_p &\le 2C_2 \sum_j |z_j|\; \| \omega_k(f;.+jh+hG) \|_p \\ &= 2C_2 \sum_j |z_j|\; \tau_k(f;hG)_p \le (\, 2C_2C_1\,)\; \tau_k(f;hG)_p . \ \square \end{aligned}
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REFERENCES

1. C. de Boor and R. DeVore. Approximation by smooth multivariate splines. Trans. Amer. Math. Soc. 276 (1983), 775-788.

2. C. de Boor and K. Höllig. B-splines from parallelepipeds. J. d'Anal. Math. 42 (1982/83), 99-115.

3. W. Dahmen and C.A.Micchelli. Recent progress in multivariate splines. in: Approximation Theory IV (C. K. Chui, L. L. Schumaker and J. D. Ward, Eds.), pp. 27-121. Academic Press, New York, 1983.

4. K. Höllig. Box splines. in: Approximation Theory V (C. K. Chui, L. L. Schumaker and J. D. Ward, Eds.), pp. 71-95. Academic Press, New York, 1986.

5. C. K. Chui, K. Jetter and J. D. Ward. Cardinal interpolation by multivariate splines. Mathematics of Computation 48 (1987), 711-724.

6. C. de Boor, K. Höllig and S. Riemenschneider. Bivariate cardinal interpolation by splines on a three direction mesh. Illinois J. Math. 29 (1985), 533-566.

7. H. Johnen and K. Scherer. On the equivalence of the K-functional and the moduli of continuity and some applications. Lecture notes in Mathematics 571 (W. Schempp and K. Zeller, Eds.), pp. 119-140. Springer, New York, 1977.

8. P. G. Binev and K. G. Ivanov. On a representation of mixed finite differences. Serdica 11 (1985), 259-268.

9. V. A. Popov. One-sided approximation of periodic functions of several variables. C. R. Acad. Bulgare Sci. 35 (1982), 1639-1642.

10. V. A. Popov and V. Kh. Khristov. Averaged moduli of smoothness for functions of several variables, and the function spaces generated by them. Proceedings of the Steklov Inst. Math. 1985, Issue 2, pp. 155-160.

11. K. G. Ivanov. On the behaviour of two moduli of functions II. Serdica 12 (1986), 196-203.

12. В. Х. Христов. Связь между обычным и усредненным модулями гладкости функций многих переменных. Доклады Болгарской АН 38 (1985), 175-178.

13. Бл. Сендов, В. А. Попов. Усреднени модули на гладкост. Издателство на БАН, София, 1983.

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