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ON A THEOREM OF KADEC

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1. Introduction. Kadec discussed in [6] the distribution of the points of Chebyshev alternation for the polynomial approximation of real functions on an interval or, equivalently, for even trigonometric approximation. The purpose of this paper is to sharpen the estimates of Kadec. We give two different proofs: The first one is elementary and somewhat simpler than Kadec's, the second uses a theorem of Erdös and Turán on the distribution of zeros of polynomials. Moreover, it is indicated how these results can be generalized for Chebyshev approximation on several intervals.

2. Trigonometric approximation. Let $f \in C(T)$ be a real continuous function on the circle **T**, f not a trigonometric polynomial. Let T_n^* be its trigonometric polynomial of degree $\leq n$ of best uniform approximation. There exists then on **T** a maximal (necessarily even) number $N \geq 2n+2$ of alternation points

(1) $$-\pi \leq t_1^{(n)} < \dots < t_N^{(n)} < \pi$$

where $E_n(f) := \|f-T_n\| = |f(t_k^{(n)})-T_n(t_k^{(n)})|$, $k=1,2,\dots$ and the signs of $f(t_k^{(n)}) - T_n(t_k^{(n)})$ alternate.

Let A (f,I) stand for the number of points (1) in an interval $I \subset T$. We shall study the uniformity of the distribution of the alternation points by examining the asymptotic behavior of $A_n(f,I)$. In [6], Kadec essentially proved that for each $\varepsilon > 0$ and infinitely many n,

$$(2) \qquad A_n(f,I) = \frac{|I|}{\pi}(n+1) + O(n^{1/2+\varepsilon}).$$

Estimates of this type do not necessarily hold for *all large n*. In [7] the second author has proved

Theorem 1. *There exists an analytic function f and two infinite sequences n'_j, n''_j so that for each $\varepsilon > 0$ and all sufficiently large j, all alternation points (1) of f for $n = n'_j$ are contained in the ε-neighborhood of 0, and for $n = n''_j$ they are contained in the ε-neighborhood of π.*

At the same time, (2) was improved to (4) below (this is incompletely mentioned in [7]).

Theorem 2. *For each $f \in C(T)$ there are infinitely many integers n such that for each interval I,*

$$(3) \qquad \left|A_n(f,I) - \frac{|I|}{\pi}(n+1)\right| \le \alpha \sqrt{n \log n},$$

where α is constant (independent of f, e.g. $\alpha = 5$).

The first proof is based on a lemma for the distribution of the local extrema of trigonometric polynomials:

Let S be a trigonometric polynomial of degree n+1 which has 2n+2 local extrema at points $-\pi \le x_1 < \dots < x_{2n+2} < \pi$, whose values $S(x_i)$, $i = 1,\dots,2n+2$ alternate in sign. We further assume that $|S(x_i)| \ge 1$, $i = 1,\dots,2n+2$ and $\|S\| \le L$ with $L > 1$. We want to estimate $M(S,I)$, the number of the points x_i which lie in an arbitrary interval $I \subset T$. We shall assume that I is of the form $I = T\setminus[-a,a]$ and that m, $0 < m < n$, satisfies $L^{1/m} < 1 + \Delta$, where $\Delta := 1 - \cos a$. For this m, there is an a_1 satisfying $\cos a_1 = L^{1/m} - \Delta$. It follows that $0 < a_1 < a$.

Lemma 3. *For m and a, as described above, we have*

$$(4) \qquad M(S,I) \le \frac{|I|}{\pi} n + \left(2 - \frac{|I|}{\pi}\right) m + \frac{2}{\pi}(a-a_1)(n-m)+5.$$

Proof. We compare S with the trigonometric polynomial of degree n, $U(t) := \sin((n-m)t)(\Delta+\cos t)^m$. We count the number of zeros of $S - U$. There are altogether $\le 2n+2$ zeros. Since $|U(t)| < 1$ for $t \in I$ and $|S(x_1)| \ge 1$, $x_1 \in I$, there are at least $M(S,I) - 1$ zeros of $S - U$ on I.

On $J_1 := (-a_1,a_1)$, we have $(\Delta + \cos t)^m > L$ and hence the maxima of $|U(t)|$ on this interval are $> L$. There are at least $[|J_1|(n-m)/\pi]-1$ of these with $U(t)$ alternating in sign. Since $|S(t)| \le L$, $-\pi \le t \le \pi$, between any two maxima of $|U(t)|$ on J_1 we have a zero of $S - U$. It follows that

$$M(S,I) + \frac{|J_1|}{\pi}(n-m) \le 2n + 5.$$

Since $2\pi - |J_1| = |I| + 2(a-a_1)$, this gives (4). □

Proof of Theorem 2. It is enough to prove

$$(5) \qquad A_n(f,I) - \frac{|I|}{\pi}(n+1) \le \frac{\alpha}{2}\sqrt{n \log n}$$

for each I of the form $T \setminus [-a,a]$ with $\pi/3 \le a \le 2\pi/3$. Indeed, if (5) is true for I, it is also true for a translate of I. Hence

$$|A_n(f,I) - \frac{|I|}{\pi}(n+1)| \le \frac{\alpha}{2}\sqrt{n \log n}$$

for all intervals of length l, $2\pi/3 \le l \le 4\pi/3$. But then (3) is true for all intervals of length $\ge 2\pi/3$ and for the complement of such intervals as well.

Since $E_n(f) \to 0$, the product $\prod_{n=1}^{\infty} (E_{n+1}(f)/E_n(f))$ diverges to 0; hence the series $\Sigma\, (E_n(f)-E_{n+1}(f))/(E_n(f)+E_{n+1}(f))$ diverges. Thus, for fixed $\varepsilon > 0$, there exist infinitely many n such that

$$(6) \qquad \gamma_n^{-1} := E_n(f)-E_{n+1}(f) \ge \frac{1}{n^{1+\varepsilon}} (E_n(f)+E_{n+1}(f)).$$

For any of these n we define the polynomial $S := \gamma_n (T_{n+1}-T_n)$. We have sign S = sign$(f-T_n)$ at each alternation point of $f-T_n$. Hence, S has 2n+2 local extrema; they alternate in sign and have absolute value ≥ 1. If we take $L_n := n^{1+\varepsilon}$ and $m := [\sqrt{n \log n}]$ then $L_n^{1/m} \to 1$, $n \to \infty$, and therefore the conditions of Lemma 3 are satisfied for n sufficiently large. This gives

$$(7) \qquad A_n(f,I) \le M(S,I)+2 \le \frac{|I|}{\pi}n + \left(2 - \frac{|I|}{\pi}\right) m + \frac{2}{\pi}(a-a_1)n + 7.$$

Now $a \in [\pi/3, 2\pi/3]$ and a_1 converges to a as $n \to \infty$; hence, for n sufficiently large,

$$(\sin \tfrac{\pi}{3})(a-a_1) \le \cos a_1 - \cos a = L_n^{1/m} - 1$$
$$\le (1+2\varepsilon) \sqrt{(\log n)/n}.$$

When this is used in (7), we get

$$A_n(f,I) - \frac{|I|}{\pi}(n+1) \le (4/3 + 4/(\pi\sqrt{3}) + \delta) \sqrt{n \log n}$$

for infinitely many n, where $\delta > 0$ can be chosen arbitrarily. Hence (3) is true for any number α greater than $8/3 + 8/(\pi\sqrt{3}) = 4{,}136 \ldots$. □

Corollary 4. *We compare the alternation points* (1) *of* T_n *with the equidistributed points* $s_k^{(n)} = \pi(k-1)/(n+1) - \pi$, $k=1,\ldots,2n+2$. *For infinitely many* n, $N=2n+2$ *and* $t_k^{(n)}$ *is approximately* $s_k^{(n)}$:

$$(8) \qquad |t_k^{(n)} - s_k^{(n)}| \le \text{const}\, \sqrt{(\log n)/n}, \quad k=1,\ldots,2n+2.$$

Indeed, the proof of Theorem 2 shows that (3) holds for infinitely many n with $T_{n+1} \neq T_n$, hence $N=2n+2$. Since (3) is valid uniformly for all subintervals I of T, we can take $I = [-\pi, t_k^{(n)})$. On one hand, I contains $k-1 = (s_k^{(n)}+\pi)(n+1)/\pi$ points $t_i^{(n)}$. On the other hand, this number is $(t_k^{(n)}+\pi)(n+1)/\pi + \delta\sqrt{n \log n}$ with $|\delta| \le \alpha$. □

A study of the distribution of the alternation points is strongly related to a study of the distribution of the zeros of the polynomials $T_{n+1}-T_n$, since, in the crucial case $T_n \neq T_{n+1}$, the zeros of $T_{n+1}-T_n$ separate the points $t_i^{(n)}$. Thus it is natural to apply here a theorem of Erdös and Turán on the distribution of zeros of polynomials to obtain an alternative proof of Theorem 2.

Theorem of Erdös and Turán. *Let* P(z) *be an algebraic polynomial with all zeros on the unit circle,*

$$P(z) = \prod_{\nu=1}^{n} (z - e^{it_\nu}), \quad -\pi \le t_\nu < \pi.$$

If Z(P,I) *denotes the number of* t_ν *in the interval* $I \subset T$ *then*

$$(9) \qquad \left|Z(P,I) - \frac{|I|}{2\pi} n\right| \le c\sqrt{n \max\{\log|P(z)| : |z| = 1\}}$$

where the constant c *is independent of* P.

In [3] Erdös and Turán proved this theorem with $c = 16$, Ganelius [5] obtained the better constant $c = 2,61...$.

<u>Alternative proof of Theorem 2.</u> We consider an n with $E_{n+1}(f) < E_n(f)$. Then $T_{n+1}(t)-T_n(t) = a_{n+1}\cos(n+1)(t-t_o)+\tilde{T}_n(t)$, where $-\pi \leq t_o < \pi$ and $\tilde{T}_n(t)$ is a trigonometric polynomial of degree $\leq n$ and

$$|a_{n+1}| = \|f-(T_n+\tilde{T}_n) - (f-T_{n+1})\|$$

$$\geq E_n(f) - E_{n+1}(f) > 0.$$

The difference $T_{n+1}-T_n$ can be written as

$$T_{n+1}(t) - T_n(t) = b_{n+1}\, z^{-(n+1)}\, P(z)$$

where $z = e^{it}$, $P(z)$ is an algebraic polynomial of degree $2n+2$ with leading coefficient 1 and b_{n+1} is a complex number satisfying $|b_{n+1}| = |a_{n+1}|/2$. Now, t is a zero of $T_{n+1} - T_n$ if and only if $z = e^{it}$ is a zero of P. Hence all zeros of P are on the unit circle. Moreover, as in (6),

$$\|P\| \leq 2\,\frac{E_n(f)+E_{n+1}(f)}{E_n(f)-E_{n+1}(f)} \leq 2\, n^{1+\varepsilon}$$

for infinitely many n. Thus the theorem of Erdös and Turán yields to the inequality (3) with $\alpha = \sqrt{2}\, c$, where c is the numerical constant of (9). □

For example, the constant of Ganelius [5], $c = 2,61...$, would lead to $\alpha = 3,70$ ... which is sligthly better than the constant in the first proof.

3. Algebraic polynomial approximation. Let $f \in C[-1,1]$ be not an algebraic polynomial, p_n its algebraic polynomial of degree $\le n$ of best uniform approximation, $E_n(f)$ the error of approximation. Then there exist a maximal number $N \ge n+2$ of points x_i of Chebyshev alternation,

$$-1 \le x_1^{(n)} < x_2^{(n)} < \ldots < x_N^{(n)} \le 1$$

where $E_n(f) := \|f-p_n\| = |f(x_i^{(n)}) - p_n(x_i^{(n)})|$ and the signs of $f(x_i^{(n)}) - p_n(x_i^{(n)})$ alternate.

Again, let $A_n(f,I)$ denote the number of points $x_i^{(n)}$ in an interval $I \subset [-1,1]$. If we use the Chebyshev measure $d\mu = (1-x^2)^{-1/2}/\pi \, dx$ a formula similar to (4) holds.

Corollary 5. *For each* $f \in [-1,1]$ *which is not an algebraic polynomial there are infinitely many integers* n *such that for each interval* $I \subset [-1,1]$,

$$(10) \qquad |A_n(f,I) - \mu(I)(n+2)| \le \alpha \sqrt{n \log n}.$$

The map $x = \cos t$ transforms f into an even function $g(t) = f(\cos t)$, p_n into an even trigonometric polynomial $T_n(t) = p_n(\cos t)$, a set $e \subset [-1,1]$ of measure μe into a set $e_1 \subset [0,\pi]$ of Lebesgue measure $me_1 = \mu e$, and an alternation point of f and p_n into two alternation points of g and T_n, symmetric with respect to 0. □

Let us replace the interval $[-1,1]$ by a finite union E of compact real intervals and let again $A_n(f,I)$ be the number of alternation points in I. Let μ_E denote the equilibrium distribution of the compact set E, then W.H.J. Fuchs [4] has generalized the result of Kadec in the following way:

If f *is not a polynomial, then there exist infinitely many* n *such that for each interval* I,

(11) $|A_n(f,I) - \mu_E(I)(n+2)| = O(n^{-1/5})$.

as $n \to \infty$. *Moreover, if* I *is contained in the interior of* E *then*

(12) $|A_n(f,I) - \mu_E(I)(n+2)| = O(n^{-1/3})$.

We remark that it is possible to obtain the same sharper estimates as in (10) for this more general case. The proof of this result is based on a potential theoretical approach to the Erdös-Turán theorems and will appear in [1].

4. A conjecture. The Erdös-Turán estimate in (9) is sharp, i. e. it is not possible to replace the right hand side by

$$n^{\alpha} \sqrt{\max \{\log|p(z)| : |z| = 1\}}$$

where $\alpha < 1/2$. On the other hand, both proofs of Theorem 2 yield the estimate $\sqrt{n}$ in (3). Hence, we want to state the following conjecture.

Conjecture: *Theorem* 2 *is not true if* (3) *is replaced by*

$$|A_n(f,I) - \frac{|I|}{\pi}(n+1)| \leq n^{1/2-\varepsilon}, \text{ where } \varepsilon > 0.$$

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