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ON ASYMPTOTIC BEHAVIOUR OF SOME CONFORMAL STRIP MAPPINGS

Melkana A. Bracalova

One classical problem in the geometric function theory is to establish a connection between the geometry of the boundary of a simply connected domain S and the conformal mapping F of S onto a canonic domain S_0 in a neighbourhood of a boundary point z_0 ($z_0 \in \partial S$). That means to find a function F^*, which depends on the geometry of ∂S such that $\lim\limits_{z \to z_0} (F(z) - F^*(z))=0$.

This problem has been investigated by many authors like L.Ahlfors, S.Warschawski, B.Rodin, A.Gol'dberg, T.Strocik, B.Eke, A.Obrock and many others. These authors have applied different methods: the geometric - analytic method, the method of extremal length, the quasiconformal method and others. In this paper is used a quasiconformal approach to this problem and especially a new theorem obtained by the author, which generalizes the Teichmüller theorem([1] , [3]) on the behaviour of a quasiconformal mapping at a point.

We denote by S_z^α ,($\alpha \in R, \alpha \neq \pm\infty$) a horizontal parallel semistrip in z -plane (z=x+i.y) defined as follows:

$$S_z^\alpha = \left\{ z=x+iy: x > \alpha,\ 0 < y < 1 \right\} .$$

Let S_ζ^a and S_w^b be semistrips in ζ-plane ($\zeta = \xi + i\eta$) and w-plane (w=u+i.v), respectively. Let

$$g(\zeta) = u(\zeta) + i.v(\zeta) \tag{1}$$

be a quasiconformal mapping (q.c. mapping) of S_ζ^a onto S_w^b with dilatation $p(\zeta)$ and partial derivatives $g_\xi = u_\xi + i.v_\xi$, $g_\eta = u_\eta + i.v_\eta$. We introduce the following notations corresponding to g :

$$N(\xi) = \left[\int_0^1 \frac{u_\eta^2 + v_\eta^2}{u_\xi v_\eta - u_\eta v_\xi} d\eta\right]^{-1}, \quad M(\xi) = \int_0^1 \frac{u_\xi^2 + v_\xi^2}{u_\xi v_\eta - u_\eta v_\xi} d\eta \tag{2}$$

$$e(\xi_1, \xi_2) = \int_{\xi_1}^{\xi_2} [M(\xi) - N(\xi)] d\xi$$

where $\xi \in [a,\infty)$ and $\xi_2 > \xi_1 > a$. As it is well known for the dilatation $p(\zeta)$ is fulfilled the equality:

$$p(\zeta) + (p(\zeta))^{-1} = (u_\xi^2 + v_\xi^2 + u_\eta^2 + v_\eta^2)/(u_\xi v_\eta - u_\eta v_\xi) . \tag{3}$$

Theorem 1. If $p(\zeta)$ satisfies

$$\iint_{S_\zeta^a} (p(\zeta) - 1)\, d\xi d\eta < \infty \tag{4}$$

then there exists the limit:

$$\lim_{\xi \to \infty} (g(\zeta) - \zeta) = C,$$

where $C = C(a,b)$ is a real number.

Theoerem 1 is a strip variant of the Teichmüller-Belinskii theorem (see [6]). This theorem is used first in [6] in order to obtain asymptotic estimates of some conformal strip mappings.

Theorem 2. If for the q.c. mapping (1) is filfilled

$$e(\xi_1, \xi_2) \to 0, \quad \xi_2 > \xi_1, \quad \xi_1 \to \infty$$

then it holds:

$$\operatorname{Re} g(\zeta) - \int_{\xi_0}^{\xi} N(\xi)\, d\xi = C + o(1), \quad \xi \to \infty \tag{5}$$

where $C = C(\xi_0)$ is a real number ($\xi_0 > a$).

Theorem 3. If the dilatation $p(\zeta)$ of (1) satisfies (4) then it follows:

$$\xi = A + \int_{\xi_0}^{\xi} N(\xi)\, d\xi + o(1), \quad \xi \to \infty$$

where $\xi_0 > a$ and $A = A(\xi_0)$ is a real number.

Theorems 2 and 3 are derived from author's results (see [4] and [5]).

Using (3) it can be verified that:

$$\int_{\xi_1}^{\xi_2}\int_0^1 \frac{(p(\zeta) - 1)^2}{p(\zeta)} d\eta\, d\xi = e(\xi_1, \xi_2) + \int_{\xi_1}^{\xi_2} \frac{(N(\xi) - 1)^2}{N(\xi)} d\xi$$

from where immediately follows by Theorem 2 that

Corollary 1. If

$$\iint_{S_\zeta^a} \frac{(p(\zeta) - 1)^2}{p(\zeta)} d\eta\, d\xi < \infty$$

then (5) is fulfilled.

Theorem 2 as well as Corollary 1, generalizes the strip variant of the Teichmüller theorem (see [1,§6.7] , [2]), which is included in Theorem 1. Using Corollary 1 in the present paper are improved some results obtained in [6] .

Let $\varphi(x)$ be a positive function, defined in the interval $[0,\infty)$ and let $\varphi(x) \in C^2[0,\infty)$. The semistrip S in z - plane is defined by

$$S = \{z = x+i.y : x > 0,\ 0 < y < \varphi(x)\}. \tag{6}$$

By Riemann theorem there exists an unique and univalent conformal mapping of S onto S_0 (where $S_0 = S_w^0$), such that the images of the points $z = \pm i.\varphi(0)$, $z = \infty$ are the points $w = \pm i$, $w = \infty$,resp.

On the function $\varphi(x)$ are imposed some of the following restrictions (see [6])

$$\left.\begin{array}{l}\text{the total variation of } \operatorname{arctg} \varphi'(x) \text{ is bounded in the}\\ \text{interval } [0,\infty).\end{array}\right\} \tag{7}$$

$$\left.\begin{array}{l}\text{for } x > x_0 \text{ and for some } q,\ 0 < q < 1 \text{ it is fulfilled}\\ \qquad \varphi''(x).\varphi(x) \geq -q(1+\varphi'^2(x))\,(\sqrt{1+\varphi'^2(x)}+1)\end{array}\right\} \tag{8}$$

$$\int_{x_0}^{\infty} \{\varphi(x).|\varphi''(x)|^2\}.\{1+|\varphi'(x)|^5\}^{-1}\, dx < \infty. \tag{9}$$

From (8) follows (see [6]) that there exists an unique and differenntiable solution $s = s(x,y)$, $t = t(x,y)$ $(s > s_0,\ s_0 = S_0(x_0)$, $0 < t < 1)$ of the following system:

$$\left.\begin{array}{l} x = x(s,t) = f(s) + r(s)\cos(t.\lambda(s)) \\ y = y(s,t) = r(s)\sin(t.\lambda(s)), \end{array}\right\} \tag{10}$$

where

$$f(s) = s - \varphi(s)/\varphi'(s), \quad \lambda(s) = \operatorname{arctg}\varphi'(s),$$
$$r(s) = \varphi(s).\sqrt{1+\varphi'^2(s)}\,/\varphi'(s).$$

Let x_s, x_t, y_s, y_t be the partial derivativesof $x=x(s,t)$ and $y=y(s,t)$, defined in (10). Then

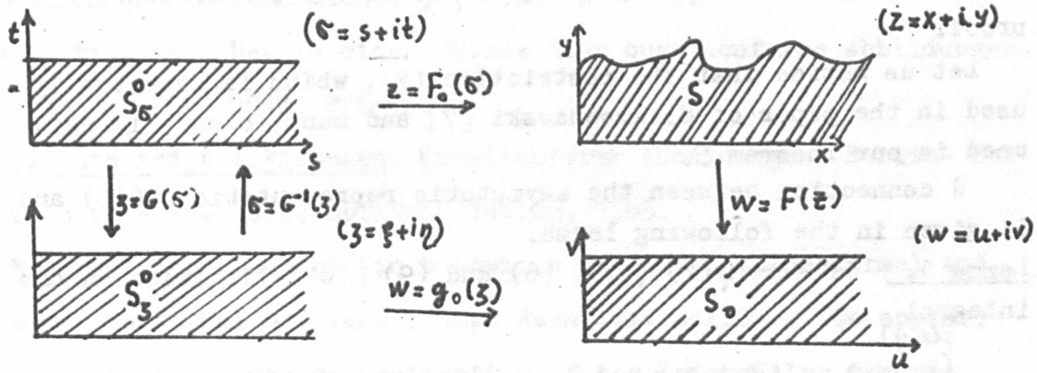
$$x_s = f' + r'\cos(t.\lambda) - tr\lambda'\sin(t\lambda),$$
$$x_t = -r\lambda\sin(t\lambda),$$
$$y_s = r'\sin(t.\lambda) + tr\lambda'\cos(t\lambda),$$
$$y_t = r.\lambda\cos(t\lambda),$$
where
$$f' = \frac{\varphi\varphi''}{\varphi'^2}, \quad r' = \frac{\varphi'^2(1+\varphi'^2) - \varphi\varphi''}{\varphi'^2\sqrt{1+\varphi'^2}} \quad \lambda' = \frac{\varphi''}{1+\varphi'^2}.$$
From here we obtain:
$$\frac{x_t^2 + y_t^2}{x_s y_t - x_t y_s} = \frac{r\lambda}{f'\cos(t\lambda)+1} = \frac{\varphi\,\mathrm{arctg}\,\varphi'}{\varphi'}\left[1 + \frac{\varphi\varphi''\left[\sqrt{1+\varphi'^2}\cos(t\lambda)-1\right]}{\varphi'^2(1+\varphi'^2)}\right]^{-1}. \quad (11)$$
Let us denote by $(s \geq s_0)$
$$\left.\begin{aligned}
\alpha(s) &= \varphi'(s)/(\varphi(s).\mathrm{arctg}\,\varphi'(s)),\\
\alpha_0(s) &= \left[\int_0^1 \frac{x_t^2 + y_t^2}{x_s y_t - x_t y_s}\,dt\right]^{-1},\\
B(s) &= \int_0^1 \left[1 + \frac{\varphi\varphi''}{\varphi'^2(1+\varphi'^2)}\left(\sqrt{1+\varphi'^2}\cos(t\lambda) - 1\right)\right]^{-1} dt .
\end{aligned}\right\} \quad (12)$$
If we denote by
$$a(s) = 1 - \frac{\varphi(s).\varphi''(s)}{\varphi'^2(s)(1+\varphi'^2(s))}, \quad b(s) = \frac{\varphi(s).\varphi''(s)}{\varphi'^2(s)\sqrt{1+\varphi'^2}}$$
then
$$B(s) = \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}}\,\mathrm{arctg}\left(\dfrac{\sqrt{a^2-b^2}.(\sqrt{1+\varphi'^2}-1)}{\varphi'.(a+b)}\right) & (a^2 > b^2)\\[2ex] \dfrac{1}{\sqrt{b^2-a^2}}\log\left|\dfrac{\sqrt{b^2-a^2}.\dfrac{\sqrt{1+\varphi'^2}-1}{\varphi'} + (a+b)}{\sqrt{b^2-a^2}.\dfrac{\sqrt{1+\varphi'^2}-1}{\varphi'} - (a+b)}\right| & (a^2 < b^2)\end{cases} \quad (13)$$
(here $a=a(s)$, $b=b(s)$, $\varphi = \varphi(s)$ and $\varphi' = \varphi'(s)$).

Using (11) and (12) we obtain that
$$\alpha_0(s) = \alpha(s).B(s),$$
where $B(s)$ is given explicitely in (13).

Let $F_0 = F_0(\sigma) = x(s,t) + i.y(s,t)$ be a q.c.mapping of S^0_σ onto S, which is defined by (10) for $s \geqslant s_0$. Its inverse map we denote by $F_0^{-1}(z)$. Wedefine a q.c.mapping $\zeta = G(\sigma)$ of S^0_σ onto S^0_ζ $(\zeta = \xi + i.\eta)$ by the equality

$$G(\zeta) = \int_0^s \alpha(p)dp \quad + i.t$$

where $\alpha(p)$ is defined in (12).The inverse map we denote by $G^{-1}(\sigma)$. Further we consider the q.c. mapping $g_0(\cdot) = F(F_0(G^{-1}(\cdot)))$, which dilatation we denote by $p_0(\cdot)$ and for which the following diagram is valid.



It can be verified, using some of the calculations in [6] and also (3) that if (8) and (9) are fulfilled then

$$\iint_{S^0_\zeta} \frac{(p_0(\zeta)-1)^2}{p_0(\zeta)}\, d\ d \quad < \infty \tag{14}$$

and if in addition also (7) is fulfilled then

$$\iint_{S^0_\zeta} (p(\zeta)-1)\, d\eta\, d\xi \quad < \infty . \tag{15}$$

In [6] is obtained the following theorem.

Theorem 4. If the function $\varphi(x)$ satisfies (7), (8) and (9) then

$$F(z) = C + \int_{s_0}^{s(x,y)} \alpha(p)dp \quad + i.t(x,y) + o(1), \; x \to \infty \tag{16}$$

uniformly with respect to y, where C is a real number.

The main result of the present paper is:

Theorem 5. If $\varphi(x)$ satisfies (8) and (9) then

$$\operatorname{Re} F(z) = C + \int_{s_0}^{s(x,y)} \alpha_0(p)dp + o(1), \; x \to \infty \tag{17}$$

uniformly with respect to y, where C is a real number.

Proof. As for the function $g_0(\zeta)$, (14) is fulfilled then by Corollary 1 follows:

$$\operatorname{Re} g \;(\quad) - \int N(p)dp = C + o(1), \; s \to \infty$$

uniformly with respect to , where C is a real number and N( ), which corresponds to the function $g_0(\quad)$ is defined in (2). After the substitution we obtain (17). This completes the proof.

Let us notice that the restriction (7), which is essentially used in the paper of S. Warschawski [7] and many others is not used in our Theorem 5.

A connection between the asymptotic representations (16) and (17) is given in the following lemma.

Lemma 1. If for $\varphi(x)$, (7), (8) and (9) are fulfilled then the integral

$$\int_{s_0}^{s(x,y)} (\alpha(p) - \alpha_0(p))\, dp \tag{18}$$

is convergent when $x \to \infty$, uniformly with respect to y.

Proof. According to (15) Theorem 3 is applicable for the function $g_0(\zeta)$ and hence there exists the limit:

$$\lim_{\xi \to \infty} \left(\xi - \int_{\xi_0}^{\xi} N(p)dp\right) = A,$$

where $A = A(\xi_0)$ is a real number and $N = N(\xi)$, which corresponds to the function $g(\zeta)$ is defined in (2). After the substitution $\zeta = G(F_0^{-1}(z))$ we obtain (18).

From this Lemma and Theorem 5 we conclude that the asymptotic representation (16) concerning the real part of F(z) follows from the asymptotic representation (17) and as it is shown below (17) is valid for a wider class of conformal mappings.

Let us consider the function φ^*, defined by

$$\varphi^*(x) = \begin{cases} \sin^3(x/\sqrt{x+1}) + 1, & x \in I_n, \; n=1,2,\dots \\ 1 & x \in R_+ \setminus \bigcup_{n=1}^{\infty} I_n, \; n=1,2,\dots \end{cases}$$

where $I_n = [2(n-1)\sqrt{n}\,\pi, (2n-1)\sqrt{n}.\pi)$.

As it can be verified for the function φ^*, (8) and (9) are fulfilled, while (7) is not. Hence for the strip mapping F(z) of the strip S, defined in (6), with $\varphi(x) = \varphi^*(x)$ we can apply Theorem 5, while Theorem 4 is not applicable.

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Institute of Mathematics with Computer Center
Bulgarian Academy of Sciences
P.O. Box 373, 1090 Sofia, Bulgaria