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A TWO-POINT PADÉ TABLE AND A PADÉ-2-TABLE

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1. Introduction. The subject of approximating a MacLaurin series and at the same time a Laurent series around $z = \infty$, using rational approximants, has received more and more attention lately. Amongst the numerous publications, those of Jones [5], Jones & Thron [6], McCabe [7], McCabe & Murphy [8], Thron [10] deserve special attention.

In a certain sense, the two-point Padé table can be seen as a type of simultaneous approximation and it is this approach that will be elaborated upon, leading to greater flexibility in the order of approximation. Use will be made of a *German-polynomial* like simultaneous approximation.

The following section contains the formulation of the most important notions, an existence theorem is included, and section 3 contains the main results. In section 4 the proofs will be given.

2. Definitions. In the sequel the attention will be focused upon two (formal) power series with complex coefficients, of the following form:

$$f_0(z) = \sum_{n=0}^{\infty} c_n z^n \ (c_0 \neq 0), \quad f_\infty(z) = \sum_{n=0}^{\infty} d_n z^{-n-1} \ (d_0 \neq 0). \tag{1}$$

It must be remarked here, that the conditions on c_0 and d_0 could be omitted, resulting in a shift with an integer power of z in the approximants; they are included for sake of simplicity.

Given two non-negative integers p and q, the problem consists of finding two complex rational functions, $R_1(z)$ and $R_2(z)$, having the same denominator Q(z), and numerators $P_1(z)$ resp. $P_2(z)$, in such a way that

$$deg\ P_1 \leq p,\ deg\ P_2 \leq q-1,\ deg\ Q \leq q, \tag{2}$$

and

$$f_0(z) - R_1(z) = O(z^{p+r+1})\ for\ z \to 0\ ,\ f_\infty(z) - z^{-1}R_2(z^{-1}) = O(z^{-q-s-1})\ for\ z \to \infty, \tag{3}$$

where the integers r and s with $r, s \geq 0$, $r + s = q$ will be taken as follows. The fact that the coefficients of Q (with a certain normalisation condition) can be used to improve the order of approximation by equating to zero the coefficients of the powers of z starting with $p+1$ for $Q(z)f_0(z)$ and q for the series $Q(z)z^{-1}f_\infty(z^{-1})$—as will become clear lateron—shows that minimising the quantity $|(p+r+1)-(q+s)|$ is the best strategy.

First we need a condition to ensure uniqueness of the solution (up to a multiplicative constant):

<u>Definition 2.1</u> The pair (f_0, f_∞) is called regular if the following determinantal conditions are satisfied:

$$H(p,q) := det \begin{pmatrix} c_p & c_{p-1} & \dots & c_{p-q+1} \\ c_{p+1} & c_p & \dots & c_{p-q+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{q+[p/2]-1} & c_{q+[p/2]-2} & \dots & c_{[p/2]} \\ d_{q-1} & d_{q-2} & \dots & d_0 \\ d_q & d_{q-1} & \dots & d_1 \\ \vdots & \vdots & \ddots & \vdots \\ d_{q+[p/2]-1} & d_{q+[p/2]-2} & \dots & d_{[p/2]} \end{pmatrix} \neq 0 \quad (p,q \geq 0). \tag{4a}$$

If, $H^+(p,q)$ denoting the determinant that arises from H(p,q) by <u>increasing all indices by 1</u>, also:

$$H^+(p,q) \neq 0 \quad (p,q \geq 0), \tag{4b}$$

the pair is called regular of full degree.
(coefficients with negative indices have to be taken 0; the block containing c's is absent for $q \leq [\frac{p+1}{2}]$)

Theorem 2.2. *Let the pair (f_0, f_∞) be regular, then there exists for each pair of non-negative integers p and q a unique pair of rational functions $R_1(z) = P_1(z)/Q(z)$ and $R_2(z) = P_2(z)/Q(z)$ with $Q(0) = 1$, satisfying the degree restrictions:*

$$deg\ P_1 \leq p, \ deg\ P_2 \leq q-1, \ deg\ Q \leq q,$$

and the following conditions on the order of approximation:
A. For $p \geq 2q$:

$$f_0(z) - R_1(z) = O(z^{p+1}) \ for\ z \to 0\ , \ f_\infty(z) - z^{-1} R_2(z^{-1}) = O(z^{-2q-1}) \ for\ z \to \infty, \tag{5}$$

B. For $p \leq 2q-1$:

$$p = 2m \quad : f_0(z) - R_1(z) = O(z^{q+m+1})\ , \ f_\infty(z) - z^{-1} R_2(z^{-1}) = O(z^{-q-m-1}), \tag{6a}$$

$$p = 2m+1: f_0(z) - R_1(z) = O(z^{q+m+1})\ , \ f_\infty(z) - z^{-1} R_2(z^{-1}) = O(z^{-q-m-2}), \tag{6b}$$

Moreover, if (4b) is also satisfied, the Q's have maximal degree: deg Q=q.

In view of this theorem we have:

<u>Definition 2.3</u> The 2-point-2-table for a regular pair (f_0, f_∞) is a table of double entry (p,q), with p and q non-negative integers, where at each point (p,q) is located the unique pair of rational functions (R_1, R_2) following from theorem 2.2.

<u>Remark 2.4</u>

a) In case A of the theorem, the coefficients of Q are determined by the function f_∞ alone: P_2/Q can be interpreted as the $[q-1/q]$ Padé approximant to $z^{-1} f_\infty(z^{-1})$ and P_1 follows from the partial sums of $Q(z) f_0(z)$.

b) For $p = 2q$ the conditions (5) and (6a)—written down for $p = 2q$—are the same: we find the same denominator in both cases, the only difference lies in the polynomials P_1.

c) In case B, both series are used to calculate the coefficients of Q, except in the case $p = 2q-1$ where the denominator from A (with $p = 2q$) is recovered. In the 2-point-2-table a Padé approximation for one function (viz. $z^{-1} f_\infty(z^{-1})$) is glued to a problem of simultaneous rational approximation using an approach dating back to Ch. Hermite (for a short historical overview: see [2]).

d) From (6a,b) we see that the order of approximation is improved, comparing with ordinary 2-point approximation. Lateron it will turn out, that this goes at the cost of an increase in the length of the recurrence relations that can be used to calculate the entries in the table.

e) The denominators Q(z), calculated for sequences of points $\{(p,k); k \geq 1\}$ with $p = 0, 1$ or 2 shows a behaviour like that of the orthogonal polynomials defined by van Iseghem [4]. The difference, however, lies in the fact that here the parity of p decides in which series the order of approximation is improved first, while in [4] the parity of q is the decisive factor.

The program is now clear: the new type of table has to be studied with respect to the existence of algorithms to calculate the entries (generalised continued fractions ?), numerical aspects, the convergence problem, the block structure etc. This paper will focus its attention on just one aspect of the problem: the algorithmic structure.

A thorough investigation (for $n = 2$) of possible sequences of simultaneous approximants satisfying certain regularity conditions has been mede by H. Padé [9] for the so-called *Latin polynomials* and by the author [1] for the type of approximation considered here.

Of course one has to stay alert about what happens on the line where the ordinary Padé table for one function is glued to the simultaneous table. Due to space-limitations only very special sequences of points will be considered here: *those, in which the value of q is increased by unity at each step.*

First we have to introduce a condition on the pair of functions to be approximated that is not as strong as normality, but that will allow the definition of a so-called regular algorithm in a relatively simple way.

<u>Definition 2.5</u> The regular pair (f_0, f_∞) of full degree and its 2-point-2-table are called algorithmic if

$$det\begin{pmatrix} P_1(p_1,k;z) & P_1(p_2,k-1;z) & P_1(p_3,k-2;z) \\ P_2(p_1,k;z) & P_2(p_2,k-1;z) & P_2(p_3,k-2;z) \\ Q(p_1,k;z) & Q(p_2,k-1;z) & Q(p_3,k-2;z) \end{pmatrix} \not\equiv 0 \quad (k \geq 3), \tag{7}$$

for all triples (p_1, p_2, p_3) of non-negative integers.

For algorithmic tables (i.e. the solution to the approximation problem is unique and there is a certain type of independence for the triplets of polynomials taken from adjacent q-rows) a regular algorithm can be defined as follows:

<u>Definition 2.6</u> A sequence of points (p_k, k), with either $1 \leq k \leq k_0$ or $k \geq k_1$ $(k_0, k_1 \geq 1)$ from the 2-point-2-table for the regular pair (f_0, f_∞) is called regular if:

(i) deg Q_k = k,

(ii) the order of approximation is improved for at least one of the functions at each step,

(iii) $$det\begin{pmatrix} P_1(p_k,k;z) & P_1(p_{k-1},k-1;z) & P_1(p_{k-2},k-2;z) \\ P_2(p_k,k;z) & P_2(p_{k-1},k-1;z) & P_2(p_{k-2},k-2;z) \\ Q(p_k,k;z) & Q(p_{k-1},k-1;z) & Q(p_{k-2},k-2;z) \end{pmatrix}$$ is a monomial in z,

(iv) the P's and Q's satisfy $X_k(z) = u_k(z)\,X_{k-1}(z) + v_k(z)\,X_{k-2}(z) + w_k(z)\,X_{k-3}(z)$, where the degree of the coefficients, which must be polynomials, does not depend on k.

As there is a connection between recurrence relations of this type and generalised orthogonality (with respect to a—possibly indefinite-innerproduct) on the space of polynomials of all degrees, it is important to find all regular algorithms in the type of Padé table defined above.

<u>3. Main results</u>. The reader who is familiar with Padé approximation knows that in the "upper third" $(p \geq 2q)$ of the 2-point-2-table exist a lot of algorithms to calculate the entries where the length of the

recurrence relation is one less than in Definition 2.6 if the attention is restricted to $P_2(z)$ and $Q(z)$ only (these originate from an ordinary Padé table for one function). If, however, one wants to include the polynomials $P_1(z)$, it is necessary to consider length 4. The main result is the following:

Theorem 3.1. *For a sequence of points (p_k, k) in an algorithmic 2-point-2-table only the following choices for the p_k lead to a regular algorithm (Definition 2.6):*
A. $p_k = p+k$ for $1 \le k \le p$, p a non-negative integer (the algorithm terminates), with

$$u_k(z) = \alpha_k z + 1,\ v_k(z) = \beta_k z + \gamma_k z^2,\ w_k(z) = \delta_k z^3 \quad (\delta_k \neq 0),$$

B. $p_k = p$ for $k \ge [(p+1)/2]$ with

$$u_k(z) = \alpha_k z + 1,\ v_k(z) = \beta_k z + \gamma_k z^2,\ w_k(z) = \delta_k z^2 \quad (\delta_k \neq 0),$$

C. $p_k = p+k$ for $k \ge p$ with

$$u_k(z) = \alpha_k z + 1,\ v_k(z) = \beta_k z^2,\ w_k(z) = \gamma_k z^3 \quad (\gamma_k \neq 0).$$

As can be seen previous theorem, there is in general no possibility to walk from the upper third of the table down into the lower part, without changing the type of algorithm. Apart from several walks which lead to changes to and fro between the form given in A and that in C, there is only one case where we can proceed with one change only:

Theorem 3.2. *In an algorithmic 2-point-2-table exists, for each natural number $p \ge 3$ exactly one sequence of points starting from (p,1), in which q increases by unity at each step, and that leads to a regular algorithm both in the upper ($p \ge 2q$, form A from Theorem 3.1) and in the lower ($p \le 2q$, form C from Theorem 3.1) part of the table: the sequence $\{(p+k-1,k); k \ge 1\}$.*

<u>Remark 3.3</u> Space limitations do not permit to look into the algorithmic structure any further: other results, reminiscent of the work by Frobenius [3] do exist; these will be treated in a more voluminous paper.

<u>4.Proofs</u>. First we introduce two power series in powers of z that carry all the information given in the pair (f_0, f_∞). Consider the pair (f_1, f_2) defined by

$$f_1(z) := f_0(z) = \sum_{n=0}^{\infty} c_n z^n \ (c_0 \neq 0), \quad f_2(z) := z^{-1} f_\infty(z^{-1}) = \sum_{n=0}^{\infty} d_n z^n \ (d_0 \neq 0). \tag{8}$$

Now the conditions on the degrees of the polynomials P_1, P_2, and Q and on the orders of approximation in Theorem 2.2, can be translated into

$$deg\ P_1 \le p,\ deg\ P_2 \le q-1,\ deg\ Q \le q, \tag{9}$$

and
A. For $p \ge 2q$:

$$Q(z) f_1(z) - P_1(z) = O(z^{p+1}),\ Q(z) f_2(z) - P_2(z) = O(z^{2q}) \tag{10}$$

B. For $p \le 2q-1$:

$$p = 2m \quad : Q(z) f_1(z) - P_1(z) = O(z^{q+m+1}),\ Q(z) f_2(z) - P_2(z) = O(z^{q+m}), \tag{11a}$$

$$p = 2m+1 : Q(z) f_1(z) - P_1(z) = O(z^{q+m+1}),\ Q(z) f_2(z) - P_2(z) = O(z^{q+m+1}). \tag{11b}$$

This shows that the new 2-point-2-table consists of an "appended" table for the function f_2, glued to a type of Padé-2-table for the functions (f_1, f_2) in a special manner.

Proof of Theorem 2.2. Inserting $Q(z) = 1 + a_1 z + a_2 z^2 + \ldots + a_q z^q$ and writing down the equations for the coefficients $a_1, a_2, \ldots, a_q$ that follow from equating to zero the right coefficients in the series $Q(z) f_1(z)$ and/or $Q(z) f_2(z)$, we find square systems of homogeneous linear equations with coefficients-determinants just those appearing in the regularity condition (4a), leading to a unique solution; the conditions (4b) show that this solution satisfies $a_q \neq 0$. The polynomials P_1 and P_2 follow in the usual manner. ∎

Proof of Theorem 3.1. Not all the calculations needed for the proof will be given. Two cases, showing the method to be exploited, will be treated and it will be very simple to fill in the details for the entire proof.

The first step consists of finding three points (p_1, k), $(p_2, k+1)$, $(p_3, k+2)$ such that the determinant described in formula (7) is indeed a monomial in z. Here two choices can be made: either all three points lie in the upper third, or they all lie in the lower part of the table (the "border-crossing" case will be treated in the proof of Theorem 3.2).

The first case is rather simple: the orders of approximation are at least $p+1$ for the first function and at least $2q$ for the second one. We already know from Theorem 2.2 that the polynomial Q has exact degree q; as q is monotonically increasing, the condition (ii) for regularity is also satisfied. Estimating the maximal degree of the determinant and the minimal order, we find a condition like

$$max(p_1 + 2, p_2 + 1, p_3) = min(p_1 + 3, p_2 + 1, p_3 + 1)\,. \tag{12}$$

Inserting $p_2 = p_1 + \delta$ and $p_3 = p_2 + \epsilon$ the solution is a matter of simple graphics: $\delta = 1$ with $\epsilon = 0$ or 1 and $\delta = 2$ with $\epsilon = 0$ or 1. This gives 4 configurations of admissable triplets, given by

$$\begin{cases} (p, p+1, p+1) & \text{(a)} \\ (p, p+1, p+2) & \text{(b)} \\ (p, p+2, p+2) & \text{(c)} \\ (p, p+2, p+3) & \text{(d)} \end{cases}$$

(the q-value has been omitted, it will be $k, k+1, k+2$ respectively; the index on the p has been dropped). The next step consists of taking the second point as first point and the third as second: this starting configuration has to be made into an admissable triplet by checking the list given above.

Now (a) and (c) stop at once, while (d) leads to either (a) or (c) and stops thereafter. The candidate that is left over is (b); this one has to lead to (a) or to itself. The conclusion is obvious: the only set of 4 points in which both the three points in the beginning and the three at the end have a monomial for determinant as in the definition for regularity, and moreover is capable of having a fifth point added, is just $(p, p+1, p+2, p+3)$.

Finally application of Cramer's rule (using the fact that the table was algorithmic) leads to part A of the theorem.

The other case, all points lie in the lower part of the table, leads to more work. We now have to distinguish cases according to the parity of the p's involved. Using a 0 for an even p and a 1 for an odd p, there are 8 basic cases (the binary representations of the numbers $1, 2, \ldots, 7$ with three digits). Only one case will be treated here, leading to an equation like (12), the others will be left to the reader; all results, however, will be given.

Consider the case 110, i.e. $p_1 = 2m_1 + 1, p_2 = 2m_2 + 1, p_3 = 2m_3$. The maximal degrees and the minimal orders are as in the array below (degrees above the dotted line):

$$\begin{array}{ccc} 2m_1 + 1 & 2m_2 + 1 & 2m_3 \\ k-1 & k & k+1 \\ k & k+1 & k+2 \\ \hline k + m_1 + 1 & k + m_2 + 2 & k + m_3 + 3 \\ k + m_1 + 1 & k + m_2 + 2 & k + m_3 + 2\,. \end{array}$$

Introducing δ and ϵ by $m_2 = m_1 + \delta, m_3 = m_2 + \epsilon$ the regularity conditions (ii) and (iii) give rise to

$$max(3, 2\delta + 2, 2\delta + 2\epsilon) = min(\delta + 3, \delta + \epsilon + 3, 2\delta + \epsilon + 4), \qquad \delta \geq -1, \epsilon \geq 0. \tag{13}$$

The solution is $\delta = 0, \epsilon = 0$ or 1 and $\delta = 1, \epsilon = 0$ or 1 and 4 admissable triplets are found:

$$\begin{cases} (2m+1, 2m+1, 2m) & \text{(a)} \\ (2m+1, 2m+1, 2m+2) & \text{(b)} \\ (2m+1, 2m+3, 2m+2) & \text{(c)} \\ (2m+1, 2m+3, 2m+4) & \text{(d)}. \end{cases}$$

This time we do not have to combine the configurations found amongst themselves, but with the admissable triplets belonging to the p–sets 100 and 101! Calculating all admissable triplets for all choices of p's and combining them into sets of 4 consecutive points, having first and last triplet satisfying the regularity conditions (ii) and (iii), we find 48 quadruplets. These fall into two classes: each solution in the first class, starting with an even p say, has a companion in the other class with exactly the same relative locations of the p–values but starting at an odd value of p. Thus we have 24 quadruplets giving rise to an algorithm in accordance with (iv). They are listed below, grouped together according to the form of the recurrence relation.

Type 1: $1+\alpha z, \beta z+\gamma z^2, \delta z^2$

$$\begin{cases} (p,p,p,p) & \text{(a)} \\ (p,p,p,p+2) & \text{(b)} \\ (p,p,p,p-1) & \text{(c)} \\ (p,p,p,p+1) & \text{(d)} \\ (p,p,p+1,p) & \text{(e)} \\ (p,p,p+1,p+2) & \text{(f)}. \end{cases}$$

Type 2: $1+\alpha z, \beta z^2, \gamma z^3$

$$\begin{cases} (p,p,p+2,p+2) & \text{(a)} \\ (p,p,p+2,p+1) & \text{(b)} \\ (p,p,p+2,p+3) & \text{(c)} \\ (p,p,p+1,p+1) & \text{(d)} \\ (p,p+1,p+2,p+2) & \text{(e)} \\ (p,p+1,p+2,p+1) & \text{(f)} \\ (p,p+1,p+2,p+3) & \text{(g)} \end{cases}$$

Type 3: $1+\alpha z, \beta z, \gamma z^3$

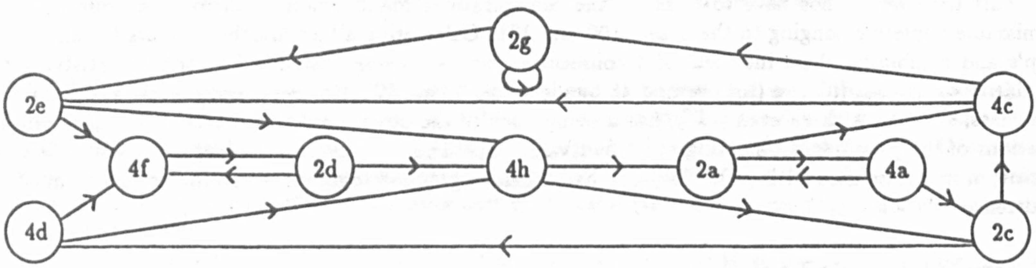
$$\begin{cases} (p,p+2,p+2,p+2) & \text{(a)} \\ (p,p+2,p+2,p+1) & \text{(b)} \\ (p,p+2,p+2,p+3) & \text{(c)}. \end{cases}$$

Type 4: $1+\alpha z, \beta z+\gamma z^2, \delta z^3$

$$\begin{cases} (p,p+2,p+2,p+4) & \text{(a)} \\ (p,p+2,p+3,p+2) & \text{(b)} \\ (p,p+2,p+3,p+4) & \text{(c)} \\ (p,p+2,p+3,p+3) & \text{(d)} \\ (p,p+1,p+1,p) & \text{(e)} \\ (p,p+1,p+1,p+2) & \text{(f)} \\ (p,p+1,p+1,p+1) & \text{(g)} \\ (p,p+1,p+1,p+3) & \text{(h)}. \end{cases}$$

The condition that the coefficients in the recurrence realation should have fixed degrees, shows that we have to combine—in general—quadruplets from the same type of algorithm only. This leads to parts B and C of the theorem. ∎

Proof of Theorem 3.2. As there is only one regular algorithm in the upper third of the table, the question of how to connect with the lower part is rather simple. It suffices to take two points in the upper part, of which the second one lies on the borderline $p = 2q$, and to look for a third point from the lower part to find all admissable "crossing" triples. Starting from $(2k+1, k), (2k+2, k+1)$ and adding $(p, k+2)$, it turns out that the only possibility is $p = 2k+3$. Calculating the form of the recurrence relation, using three points from the upper part together with this fourth point from the lower part, by Cramer's rule, we find that we still have form A from Theorem 3.1. To continue the walk in the table, it is now necessary to solve the three point problem for the borderline point and the point found above, i.e. for the points $(2m, m), (2m+1, m+1)$ and $(p, m+2)$. All the admissable triplets have been found in the proof of Theorem 3.1, and we can select, from the list of admissable quadruplets, those choices that will enable us to continue the walk through the table. Using the notation from the previous proof, the following "flow-chart" can be given, where $(2e), (2g), (4f)$ and $(4h)$ are the admissable quadruplets of entry into the lower part:



From this flow-chart it is obvious that the only possibility to continue the algorithm, at the cost of one change of type only, is as asserted in the theorem. ∎

Remark. Of course it is possible to find more sequences of points which give rise to recurrence relation with four terms, in which the types 2 and 4 are intermingled (or that show type 2 from a certain point on). To get a complete picture, however, one should make a flow-chart containing all 24 admissable quadruplets as given in Theorem 3.1, with all their interrelations.

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