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APPROXIMATIONS OF SINGULARITIES AT CORNERS WITH DIFFERENT ANGLES

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The paper refers to experiences with numerical approximation to singularities at corners and inclusion of singular solutions of boundary value problems.

1. Introduction

Singularities at corners occur frequently in linear and nonlinear elliptic boundary value problems. It is important for the numerical calculation of the solution to know the type of the singularity. The type depends in the plane strongly on the angle at the corner. This is studied on different (mostly recently calculated) examples coming from applications. Numerical results are given.

If one has "a problem of monotonic type"

(1.1) $\quad Tu = \phi(x)$

then the method of approximations gives in not too complicated cases an inclusion for the wanted solution with a reasonable work of computation. In many cases (even if singularities occur) this is the only method, which gives lower and upper bounds for the solution one can guarantee.

The method may be described on a very simple class of problems: Let B a connected domain in the x-y-plane with piecewise smooth boundary ∂B. We consider the Dirichlet-Problem for an unknown function u(x,y):

(1.2) $\quad -\Delta u = -\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = S(x,y)$ in B

(1.3) $\quad u = \Psi(x,y)$ on ∂B

with given continuous functions S, Ψ; the operator

(1.4) $\quad Tu = \{-\Delta u - S$ in B, $u - \Psi$ on $\partial B\}$

is of monotonic type, this means (compare Collatz [52], Walter [70], Bohl [74], Schröder [80], a.o.)

(1.5) $\quad Tu \le Tw$ implies $u \le w$ in B.

The sign $\le$ means the classical ordering of real numbers and the inequalities may be valid for every component.

For the simple type of problems (1.2) (1.3) the monotonicity is equivalent to the classical maximum principle, but the appicability of the monotonicity is much greater than of the maximum principle.

For getting good error bounds one approximates u by functions w of a class W of functions $w(x,y,a_1,\ldots,a_p)$ which depends on parameters a_ν. One determines there values of the a_ν from the optimization problem (compare f.i. Watson [80], Meinerdus-Merz [82] a.o.)

(1.6) $\quad -\delta \leq -\Delta w - S \leq \delta \quad$ for $\quad (x,y) \in B$

(1.7) $\quad -\hat{\delta} \leq w - \Psi \leq \hat{\delta} \quad$ for $\quad (x,y) \in \partial B$

(1.8) $\quad \delta + \gamma\,\hat{\delta} = \text{Min.}$

If possible one chooses the class W in such a way that either w satisfies the differential equation, $\delta = 0$ or the boundary condition $\hat{\delta} = 0$; if none of this is possible, one chooses γ as fixed positive constant.

If the optimization problem (1.6)(1.7)(1.8) is linear, one discretizes the problem and can use wellknown procedures for finite linear programming on a computer.

The main problem is a suitable choice of the function class W in the case that singúlarities occur. The paper deals with corners of different angles in the plane.

2. Corner-singularities

We consider a corner on the boundary ∂B with the angle α of the tangents of ∂B at Q fig. 1, with $0 < \alpha \leq 2\pi$.
We introduce polar coordinates r,φ as in fig.1; the potential function

(2.1) $\quad p = r^{m\pi/\alpha} \sin(\frac{m\pi}{\alpha}\varphi) \quad (m = 1,2,\ldots)$

satiesfies $\Delta P = 0$ and vanishes along the tangents of ∂B at Q.

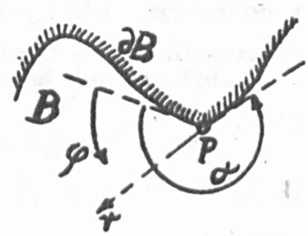

Fig. 1

One has "entering" corners for $\alpha > \pi$ and "reentering" corners for $0 < \alpha < \pi$. For $\alpha = \pi$ there is no jump in the direction of the tangent; $\alpha = 2\pi$ gives a "slit", fig. 2.
The function P of (2.1) is regular for

(2.2) $\quad \alpha = \pi/q$ for $q = 1,2,3,\ldots$

and reduces than to a polynomial.

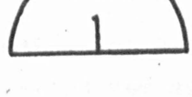
Fig. 2

In spite of this the solution u of the boundary problem (1.2)(1.3) can be singular at a corner with angle $\alpha = \pi/q$. For instance the torsion problem for a beam of cross-section B as in fig. 3 is described by

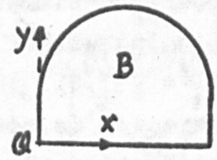

Fig. 3

$$(2.3)\qquad -\Delta u = 1 \quad \text{in} \quad B$$
$$u = 0 \quad \text{on} \quad \partial B\,.$$

In P we have $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = 0$, but Δu has the limit value 1 by approximating the point P.

Other types of singularites as discontinuities in the given boundary values, logarithmic terms in ringdomains, fig. 4, need other types of approximating singular functions, (Collatz, Grothkopf Hayman [87]), Kondrat'ev [67] suggests the use of terms

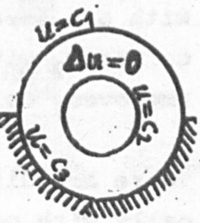

Fig. 4

$$(2.4)\qquad z^{\mu}(\ln z)^{\nu} \text{ with } z = re^{i\varphi},$$
and μ,ν real constants.

Many problems of the type (1.2)(1.3) with corners have been treated numerically on computer (compare f.i. Collatz [81],[86],[87]); we mention a far examples

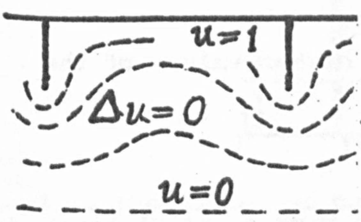

Fig. 5

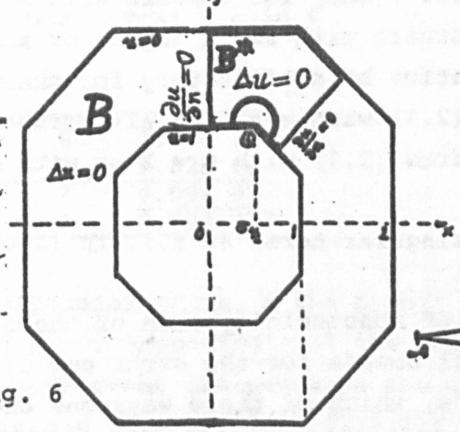

Fig. 6


Fig. 7

1) ideal flow of a river with walls, fig. 5 u=const are the streamlines.
2) the ringdomain, fig. 6, temperature in a chimney, compare Schwarz [86], p.483.
   In this example we have at the inner boundary corners with an angle $\alpha = {}^{5\pi}/_{4} > \pi$, f.i. at the corner Q and the coresponding function P in (2.1) is not univalued in the double connecting domain B and cannot be used. But if there are symmetries in B f.i. reflections as in fig.6, one can calculate the wanted function in the bounded simple connected domain B* fig.6.
3) Oscillations of a champed rhombic membran, fig. 7, are described by

$$(2.5)\qquad -\Delta u = \lambda u \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B$$

with the domain

$$B = \{(x,y),\ |y| < (1-|x|)\ \tan\alpha\}.$$

Goerisch-Zimmermann [86] used for $\alpha = \pi/24$ (or 7.5°) polynomials beginning with $r^{12}\cos(12\ \varphi)$ (polarcoordinates r,φ counted at the corner $P_1 = (-1,0)$. The angle at the corner $P_2 = (0, \tan(\pi/24)$ is $\beta = 11\pi/12$; here π/β is not an integer. The numerical results are very good even by neglecting the terms with β, because the authors used polynomials of very high degree.

4) The problem (2.3) of distribution of temperature u with the domain B, fig.8, was originally calculated with respecting only the corner P_1, with $\alpha = 5\pi/4$; but by respecting also the corner P_2 with $\beta = 3\pi/4$ the results could be improved, Collatz [86].

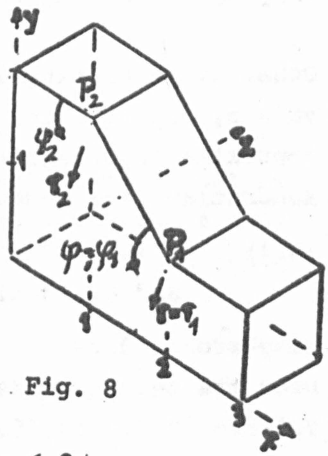

Fig. 8

There are different ways in dealing numerically with corners:

A) Using terms (2.1) only for corners with $\pi < \alpha \leq 2\pi$.

B) Neglecting corners with small angles or angles near to π. This may sometime be satisfactory for small accuracy.

C) Using terms (2.1) wirh $m = 1$ at all corners.

C) Using also terms (2.1) with $m = 2$ or with higher values of the integer m.

E) Using other singular terms as f.i. in (2.4)

<u>Thesis:</u> In case of monotonicity each of these ways A) to E) gives numerical bounds for the error one can guarantee. One has to decide, which of these ways one chooses, which price (amount of work) one is willing to pay for a wanted accuracy.

3. Test-Problem

We consider the torsion problem

$$(3.1) \qquad -\Delta q = 1 \text{ in } B$$
$$q = 0 \text{ on } \partial B$$

for a beam of cross-section B, where the domain B is given by Fig. 9, with x,y as two symmetry-axis. We have at the corner S_1 the angle $\beta = 7\pi/36$ or 35° and at the corner Q_1 the angle $\alpha = 47\pi/36$ or 235°.

Introducing $q = -\frac{1}{4}(x^2+y^2) + u(x,y)$ we get

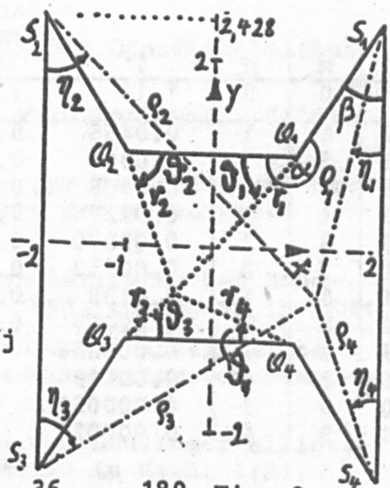

Fig. 9

(3.2) $\Delta u = 0$ in B

$$u = \frac{1}{4}(x^2+y^2) \quad \text{on} \quad \partial B$$

We approximate u by

$$u \approx v = \sum_{j=1}^{\ell} a_j p_j + \sum_{j=1}^{m} b_j s_j + \sum_{j=1}^{n} c_j t_j$$

with

$$p_j = \text{Re}\,[(x+iy)^{2(j-1)}]$$

$$s_j = \sum_{k=1}^{4} r_k^{\alpha_j} \cdot \sin(\alpha_j \theta_k), \qquad \alpha_j = j \cdot \frac{36}{47} = j \cdot \frac{180}{235}$$

$$t_j = \sum_{k=1}^{4} \rho_k^{\beta_j} \cdot \sin(\beta_j \eta_k), \qquad \beta_j = j \cdot \frac{36}{7} = j \cdot \frac{180}{35}$$

Here are r_k, θ_k polarcoordinates at the corner Q_k and ρ_k η_k at the corner s_k.

One gets the error bounds $|v-u| < \delta$ in B

ℓ	m	n	δ
1	0	0	1.11
3	4	1	0.022 68
4	8	2	0.002 97
10	9	4	0.000 022

ℓ is the number of polynomial terms, m the number of singularity-terms at the corners Q_k and n correspondingly at the corners S_k; we observe that all these terms have great influence on the accuracy. We have selected some combinations (ℓ,m,n); f.i. (3,4,1); the accuracy is remarcable better compared with every neighbour-combination (2,4,1), (3,3,1) or (3,4,0) with smaller numbers of terms. Using more terms with (4,4,1) (3,5,1) or (3,4,2) gives only a small improvement. The combinations (4,8,2) (10,9,4) show the similar behaviour which one can observe also with other problems. The table gives also approximate values for the function u at the origin (0,0).

ℓ	m	n	δ	approximate value for u(0,0)
1	0	0	1.11	1.36199
2	4	1	0.0455	0.4536
3	3	1	0.0804	0.5132
3	4	0	0.0933	0.4473
3	4	1	0.0227	0.4576
3	8	2	0.00970	0.4570
4	7	2	0,00953	0.4571
4	8	1	0.0135	0.4692
4	8	2	0.00297	0.4680
9	9	4	0.000056	
10	8	4	0.000194	
10	9	3	0.000059	
10	9	4	0.000022	

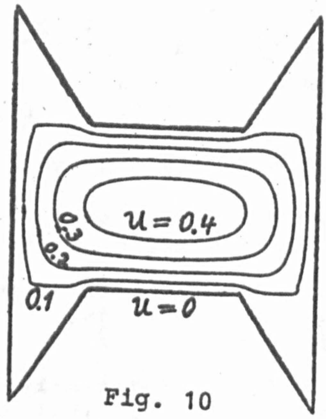

Fig. 10

Fig. 10 gives some contour lines u = 0.1, 0.2, 0.3, 0.4.

Many open questions are in the field of approximation of singularities and much research and experience is necessary, even for nonlinear problems.

Threedimensional singularities are usually much more difficult (compare f.i. Dobrowolski [85], Tolksdorf [83], Whiteman [84]); a paper about these problems is prepared.

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