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ON THE APPROXIMATION OF THE DERIVATIVES OF P.V. INTEGRALS

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1. Introduction. Let $w(x)=\psi(x)u^{\alpha,\beta}(x)$, $x\in I:=[-1,1]$ be a generalized Jacobi weight, where $u^{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$, $\alpha,\beta>-1$ and $\psi>0$, is a continuous function satisfying a "Dini type" condition, i.e. $\int_0^1\omega(\psi;\delta)\delta^{-1}d\delta<\infty$, where $\omega(\psi;\ )$ denotes the modulus of continuity of the function ψ.
Then, we denote by $\Phi(wf;t)$ the integral in the Cauchy principal value sense of the function f, associated with the weight w and defined by

$$\Phi(wf;t) = \lim_{\varepsilon\to 0^+}\int_{|x-t|\geq\varepsilon}\frac{f(x)}{x-t}w(x)dx \quad , \quad -1<t<1 .$$

Throughout this paper $\{p_m(w)\}_{m\in N}$ denotes the sequence of the orthonormal polynomials on I associated with the weight function w; hence

$$p_m(w;x) = \alpha_m(w)x^m + \text{lower degree terms} \quad , \quad \alpha_m(w)>0 ,$$

and

$$\int_{-1}^1 p_m(w;x)p_n(w;x)w(x)dx = \delta_{m,n} .$$

The zeros of $p_m(w)$, denoted by $x_{m,k}=\cos\theta_{m,k}$, are ordered so that $0<\theta_{m,m}<\ldots<\theta_{m,1}<\pi$. Finally, let $\lambda_{m,k}$ be the Christoffel constants.

If the function f is sufficiently smooth (differentiable, for instance), then the integral $\Phi(wf;t)$ can be "well" approximated by

$$(1) \qquad \Phi_m(wf;t) = A_m(t)f(t) + \sum_{k=1}^{m}\frac{\lambda_{m,k}}{x_{m,k}-t}f(x_{m,k}) \quad ,$$

with

$$A_m(t) = \Phi(w;t) - \sum_{k=1}^{m}\frac{\lambda_{m,k}}{x_{m,k}-t} .$$

Indeed, if $E_m(wf;t) = \Phi(wf;t) - \Phi_m(wf;t)$ denotes the remainder term, the authors have proved in [1] that the following bound

$$|E_m(wf;t)| \leq \text{const } m^{-k}\omega(f^{(k)};m^{-1})\log m \quad , \quad f\in C^k(I) \quad , \quad k\geq 1 \quad ,$$

holds on every closed interval Δ such that $\Delta\subset(-1,1)$.
Nevertheless, if the function f is only Hölder continuous, then the sequence $\{\Phi_m(wf;t),\ m\in N\}$ defined by (1) does not converge to $\Phi(wf;t)$ almost everywhere in $(-1,1)$. For example, in the case $w(x)=(1-x^2)^{-\frac{1}{2}}$, if $t=\cos\pi\xi$ with ξ irrational, then the above mentioned sequence does not converge for the Hölder continuous function $|x-t|^{\lambda}$, $0<\lambda<\frac{1}{2}$ (see [8]).
Yet, we have recently proved in [1] the existence of subsequences convergent uniformly in every closed set Δ enclosed in $(-1,1)$ for functions f which are not very smooth. Indeed, denoting by $x_{c(m)}$ the closest knot to t, defined by $x_{m,k}\leq t\leq x_{m,k+1}$, $|t-x_{c(m)}|=\min\{t-x_{m,k},x_{m,k+1}-t\}$ and introducing the set $\tilde{N}=\{m\in N \ / \ |\theta_{c(m)}-\theta|\geq \text{const } m^{-1}\}$, where $x_{c(m)}=\cos\theta_{c(m)}$, $t=\cos\theta$, we have proved the following
Theorem 1.1 . The set $\tilde{N}$ is infinite, i.e. $|\tilde{N}|\sim|N|$, and if the continuous function f is such that $\int_0^1 \omega(f;\delta)\delta^{-1}d\delta<\infty$, then the sequence $\{\Phi_m(wf;t),\ m\in\tilde{N}\}$ converges to $\Phi(wf;t)$ uniformly on any closed set $\Delta\subset(-1,1)$.
Now, we point out that the approximation $\Phi_m(wf;t)$ of $\Phi(wf;t)$ may be used in solving singular integral equations (SIE) with a collocation method. The uniform convergence of the approximation of the integral $\Phi(wf;t)$ in closed sets enclosed in $(-1,1)$ generally is not sufficient to assure the convergence of the collocation method; for instance, this happens when the collocation points are zeros of orthogonal polynomials in I. Therefore, it is important to study the uniform convergence of the same approximation on the whole interval $(-1,1)$.
Moreover, we recall that the derivative $\frac{d}{dt}\Phi(wf;t)$ appears in some integrodifferential equations concerning several branches of physics and engineering [2,3,6]. Further, the analytic solution of the integral equations with logarithmic singularities in the kernel may be represented by the derivatives of Cauchy principal value integrals [4].
In this paper we generalize the results of the previous theorem in at least one direction. In fact, we study the convergence of the

sequence of the derivatives $\{\frac{d^p}{dt^p}\Phi_m(wf;t),\ m\in\tilde{N}\}$ to $\frac{d^p}{dt^p}\Phi(wf;t)$ on the whole interval $(-1,1)$, obtaining results which are more general and careful than those proved in [5] as far as regards the hypotheses on the function f and on the weight w.

2. On the convergence of the sequence $\{\frac{d^p}{dt^p}\Phi_m(wf;t);\ m\in\tilde{N}\}$.

To obtain the proof of the main result of the present section, the following lemmas are needed.

Lemma 2.1 . If $f\in C^r(I)$, $r\geq 0$, then for each $m\in N$ there exists a polynomial t_m of degree $m\geq 4(r+1)$ such that

$$|f^{(k)}(x)-t_m^{(k)}(x)|\leq \text{const}\,[m^{-1}\sqrt{1-x^2}]^{r-k}\omega(f^{(r)};m^{-1}\sqrt{1-x^2}),\quad 0\leq k\leq r,\ x\in I,$$

$$|t_m^{(p)}(x)|\leq \text{const}\,[\Delta_m(x)]^{r-p}\omega(f^{(r)};\Delta_m(x)),\quad p>r,\ x\in I,$$

where $\Delta_m(x)=\max\{m^{-1}\sqrt{1-x^2},\ m^{-2}\}$.

Lemma 2.1 can be found in [9].

If we introduce the class of functions $DT^{(p)}(A):=\{f\in C^p(A)/\int_0^1\omega(f^{(p)};\delta)\delta^{-1}d\delta<\infty\}$, where $A\subseteq I$, $p\geq 0$, then we can state the following

Lemma 2.2 . If $w(x)=\psi(x)u^{\alpha,\beta}(x)$, $0<\psi\in DT^{(0)}(I)$, $\alpha,\beta>-1$, then for every function $f\in DT^{(p)}(I)$, $p\geq 1$, the integral $\int_{-1}^{1}\frac{d^p}{dt^p}\left[\frac{f(x)-f(t)}{x-t}\right]w(x)dx$ exists and the identity

$$\frac{d^p}{dt^p}\int_{-1}^{1}\frac{f(x)-f(t)}{x-t}w(x)dx=\int_{-1}^{1}\frac{d^p}{dt^p}\left[\frac{f(x)-f(t)}{x-t}\right]w(x)dx,\quad |t|<1,$$

holds for $p\geq 1$.

We omit the proof which is based on known results of classical analysis and elementary inequalities for the modulus of continuity.

Lemma 2.3 . If $w(x)=\psi(x)u^{\alpha,\beta}(x)$, $0<\psi\in DT^{(0)}(I)$, then the inequalities

$$\text{const}\log m\leq\sum_{k=1}^{m}\frac{\lambda_{m,k}}{|x_{m,k}-t|}\leq \text{const}\log m\ ,\qquad \text{if}\ \alpha,\beta\geq 0\ ,$$

$$\sum_{k=1}^{m}\frac{\lambda_{m,k}}{|x_{m,k}-t|}\leq \text{const}\ u^{\alpha,\beta}(t)\log m\ ,\qquad \text{if}\ -1<\alpha,\beta<0\ ,$$

hold for $t\in(-1,1)$ and $m\in\tilde{N}$.

The proof follows easily from an analogous property proved by the authors in [1, Lemma 3.II] and from the inequalities: $\theta_{m,k+1}-\theta_{m,k} \sim m^{-1}$, $\lambda_{m,k} \sim m^{-1} u^{\alpha+\frac{1}{2},\beta+\frac{1}{2}}(x_{m,k})$, that can be found in [7].

Now, since rule (1) has degree of exactness $2m$, we have

$$\left|\frac{d^p}{dt^p} E_m(wf;t)\right| = \left|\frac{d^p}{dt^p} E_m(w(r_m-r_m(t));t)\right| \leq \left|\frac{d^p}{dt^p} \Phi(w(r_m-r_m(t));t)\right| +$$

$$+ \left|\frac{d^p}{dt^p} \Phi_m(w(r_m-r_m(t));t)\right| ,$$

where $r_m=f-t_m$, t_m being the polynomial of Lemma 2.1.

Finally, using the previous lemmas and omitting the calculations, one can deduce the following

Theorem 2.4 . Let $w(x)=\psi(x)u^{\alpha,\beta}(x)$, $0<\psi\epsilon DT^{(p)}(I)$; then

$$\lim_{m\epsilon N^{\sim}} \left|\frac{d^p}{dt^p} E_m(wf;t)\right| = 0 \quad , \quad \text{for } t\epsilon(-1,1), \quad p\geq 0,$$

if

i) $\alpha,\beta\geq 0$, $f\epsilon DT^{(p)}(I)$,

or

ii) $-1<\alpha,\beta<0$, $f^{(p)}\epsilon \, \mathrm{Lip}_M \lambda$, $\lambda>-\min(\alpha,\beta)$.

Moreover, the inequalities

$$\left|\frac{d^p}{dt^p} E_m(wf;t)\right| \leq \text{const } \omega(f^{(p)};m^{-1}) \log m \text{ , if } \alpha,\beta\geq 0, \ f\epsilon DT^{(p)}(I), \ p\geq 0,$$

$$\left|\frac{d^p}{dt^p} E_m(wf;t)\right| \leq \text{const } m^{-2(\lambda-\gamma)} \log m \text{ , if } -1<\alpha,\beta<0, \ f^{(p)}\epsilon \mathrm{Lip}_M\lambda, \ \lambda>\gamma=-\min(\alpha,\beta), \ p\geq 0,$$

hold for $t\epsilon(-1,1)$ and $m\epsilon N^{\sim}$.

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