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MEAN APPROXIMATION VIA INTERPOLATION BY TRANSLATION

Franz-Jürgen Delvos

Introduction. Interpolation of periodic functions on a uniform mesh $t_j = 2\pi j/n$ ($j \in \mathbb{Z}$) via translates of a given periodic function g has been discussed in several papers [1, 2, 3, 4, 6]. In the recent paper [2] we proved convergence of interpolation by translation in the uniform norm. In this paper we continue these investigations and derive error estimates in the mean square norm. Applications to periodic spline interpolation and rational trigonometric interpolation are given. Moreover, a variational characterization of interpolation by translation is presented.

1. Interpolation by translation.

Let $g \in C_{2\pi}$ have the absolutely convergent Fourier series

$$g(t) = \sum_{k=-\infty}^{\infty} d_k \exp(ikt)$$

with

$$d_{-k} = d_k \ , \ 0 < d_{k+1} \le d_k \le 1 \ (k \in \mathbb{N}) \ , \ d_0 \ge 0 \ .$$

For $n = 2m+1$ ($m \in \mathbb{N}$) we define the uniform mesh $t_j = 2\pi j/n$ ($j \in \mathbb{Z}$). It was shown in [1, 3] that the linear space

$$V_n^1(g) = \text{span}\{ 1, g(.-t_1)-g, \dots, g(.-t_{n-1})-g \}$$

has dimension n and is translation invariant with respect to t_1. For $k \in \mathbb{Z}-n\mathbb{Z}$ we introduce the functions

$$B_k(t) = \sum_{j=0}^{n-1} g(t-t_j)\exp(ikt_j)$$

which have the properties

$$(1) \qquad B_k(t) = n \sum_{r=-\infty}^{\infty} d_{k+rn} \exp(i(k+rn)t)$$

$$(2) \qquad B_k(t_j) = B_k(0)\exp(ikt_j) \neq 0$$

The functions $1, B_1, \ldots, B_{n-1}$ form a basis of $V_n^1(g)$ and the function M defined by

$$M(t) = (1 + \sum_{j=1}^{n-1} B_j(t)/B_j(0))/n$$

is the *periodic Lagrange function* of $V_n^1(g)$, i. e. ,

$$M(t_j) = \delta_{0,j} \qquad (0 \le j < n)$$

In view of the translation invariance of $V_n^1(g)$ there is a unique interpolant $Q_n(f) \in V_n^1(g)$ of $f \in C_{2\pi}$ which is given by

$$Q_n(f)(t) = \sum_{j=0}^{n-1} f(t_j)M(t-t_j)$$

and which satisfies $Q_n(f)(t_j) = f(t_j)$ ($j \in \mathbb{Z}$) .

The interpolant $Q_n(f)$ is also given by (see [2])

$$(3) \qquad Q_n(f) = c_{0,n}(f) + \sum_{k=1}^{n-1} c_{k,n}(f)\, b_k$$

with

$$(4) \qquad b_k(t) = B_k(t)/B_k(0)$$

and

$$c_{k,n}(f) = \frac{1}{n} \sum_{j=0}^{n-1} f(t_j)\exp(-ikt_j)$$

For notational convenience we put $e_k(t) := \exp(ikt)$, $k \in \mathbb{Z}$. The inner product on $L^2_{2\pi}$ is given by

$$(f,g) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt \quad .$$

Note that

$$Q_n(e_{k+sn}) = b_k \quad , \quad Q_n(e_{-k}) = b_{n-k} = \overline{b_k}$$

for $s \in \mathbb{Z}$ and $0 < k < n \doteq 2m+1$. It follows from (4) and (1) that the following *orthogonality relations* hold :

$$(5) \qquad (Q_n(e_j),Q_n(e_k)) = 0 \qquad (-m \le j,k \le m \ , \ j \neq k) \ ,$$

$$(6) \qquad (e_j-Q_n(e_j),e_k-Q_n(e_k)) = 0 \qquad (-m \le j,k \le m \ , \ j \neq k) \ .$$

Recall that $Q_n(e_0) = e_0$. It follows from (1) that

(7) $Q_n(e_j) \quad = \quad \sum_{r=-\infty}^{\infty} (d_{j+rn}/d_{j,n}) e_{j+rn} \qquad (\ n = 2m+1\ )$

with

(8) $d_{j,n} \quad = \quad \sum_{r=-\infty}^{\infty} d_{j+rn}$

This implies

(9) $\|Q_n(e_j)\| \ \le \ 1 \quad (\ -m \le j \le m\ )$

Taking into account (3) and (5) we obtain

(10) $\|Q_n(f)\|^2 \ \le \ \sum_{k=-m}^{m} |c_{k,n}(f)|^2 \qquad (\ n = 2m+1\ )$

It was shown in [2] that

$$D_n \quad = \quad \sum_{r=1}^{\infty} d_{rm} \qquad (\ n = 2m+1\ )$$

is an appropriate number to describe the *approximation power* of Q_n

In particular we have

(11) $0 \ \le \ d_{j,n} - d_j \ \le \ D_n \qquad (\ -m \le j \le m\ )$

Proposition 1 *Let* $0 < |j| \le m$ *and* $n = 2m+1$. *Then we have*

(12) $\|e_j - Q_n(e_j)\| \ \le \ \sqrt{2}\, D_n/d_j$

Proof. Using (11) and (7) we get

$$\|e_j - Q_n(e_j)\|^2 \ = \ (\ (d_{j,n} - d_j)^2 + \sum_{r\neq 0} (d_{j+rn})^2\ )/(d_{j,n})^2 \ \le \ 2\, D_n^2/d_j^2 \ .$$

2. The approximation power of interpolation by translation. Using the Fourier coefficients of g we define the linear operator A by

$$Af \quad = \quad \sum_{k\neq 0} (f, e_k) d_k^{-1} e_k$$

where $f \in dom(A) = \{\ g*h : h \in L^2_{2\pi}\ \}$. Note that $\|Af\| < \infty$.

We denote by F_m the Fourier partial sum projector :

$$F_m(f) \quad = \quad \sum_{j=-m}^{m} (f, e_j) e_j$$

It follows from $F_m A = AF_m$ that

(1) $\|f - F_m(f)\| \ \le \ d_m \|A(f - F_m(f))\| \qquad (\ f \in dom(A)\ )$

Proposition 2 *Let* $f \in ran(F_m)$ *and* $n = 2m+1$. *Then we have*

$$(2) \qquad \|f - Q_n(f)\| \le \sqrt{2}\, D_n \|Af\|$$

Proof. Using Proposition 1 and (6) of section 1 we get

$$\|f-Q_n(f)\|^2 \le \sum_{j=-m}^{m} |(f,e_k)|^2 \|e_j - Q_n(e_j)\|^2$$

$$\le 2 D_n^2 \sum_{j=-m}^{m} |(f,e_j)/d_j|^2 = 2 D_n^2 \|Af\|^2$$

Proposition 3 *Assume* $f \in dom(A)$ *and* $n = 2m+1$ *Then we have*

$$(3) \qquad \|Q_n(f-F_m(f))\| \le D_n \|A(f-F_m(f))\|$$

Proof. Put $u = f-F_m(f)$. Using (10) of section 1 we obtain

$$\|Q_n(u)\|^2 \le \sum_{k=-m}^{m} |c_{k,n}(u)|^2 = \sum_{k=-m}^{m} \Big| \sum_{r=-\infty}^{\infty} c_{k+rn}(u)\Big|^2$$

$$\le \sum_{k=-m}^{m} \Big(\sum_{r\ne 0} |(u,e_{k+rn})/d_{k+rn}|^2 \Big) \Big(\sum_{r\ne 0} (d_{k+rn})^2 \Big) \le D_n^2 \|Au\|^2 .$$

We are now in a position to prove the main result concerning the approximation power of Q_n in the L_2-norm.

Theorem 1 *Let* $f \in dom(A)$ *Then we have*

$$(4) \qquad \|f - Q_n(f)\| \le 4\, D_n \|Af\|$$

Proof. Taking into account Propositions 2 and 3 and (1) we get

$$\|f-Q_n(f)\| \le \|f-F_m(f)\| + \|F_m(f)-Q_n(F_m(f))\| + \|Q_n(f-F_m(f))\|$$

$$\le D_n\|A(f-F_m(f))\| + \sqrt{2}\, D_n\|A(F_m(f))\| + D_n\|A(f-F_m(f))\| \le 4D_n\|Af\|$$

3. Applications. As a first example we consider

$$g(t) = \sum_{k\ne 0} k^{-2s}\exp(ikt) =: P_{2s}(t)$$

with $2s > 1$ For $s \in \mathbb{N}$ $V_n^1(P_{2s})$ is the space of *periodic* splines of degree 2s-1 with uniform mesh t_j, $j \in \mathbb{Z}$ (see [1,3,7]).

Corollary 1.1 *Assume that* $f \in C_{2\pi}$ *satisfies*

$$\|Af\|^2 = \sum_{k\neq 0} k^{4s}|(f,e_k)|^2 < \infty$$

Then we have

(1) $$\|f-Q_n(f)\| \leq 4\,\zeta(2s)\,\|Af\|/m^{2s} \quad (n = 2m+1).$$

Proof. This follows from $D_n = \zeta(2s)m^{-2s}$ [2]. For $s \in \mathbb{N}$ Corollary 1.1 was proved by Golomb [5] . It is applicable for functions $f \in C^{2s-1}_{2\pi}$ with $D^{2s}f \in L^2_{2\pi}$.

Next we consider the *rational trigonometric function* [2]

$$g(t) = 1 + 2\sum_{k=1}^{\infty} e^{-kb}\cos(kt) = \sinh(b)/(\cosh(b)-\cos(t))$$

with $b > 0$.

Corollary 1.2 *Assume that* $f \in C_{2\pi}$ *satisfies*

$$\|Af\|^2 = \sum_{k\neq 0} \exp(|k|2b)\,|(f,e_k)|^2 < \infty$$

Then we have

$$\|f - Q_n(f)\| \leq 4\,\|Af\|\,e^{-mb}/(1 - e^{-mb}) \quad (n = 2m+1).$$

Remark. Obviously, Corollary 1.2 holds for periodic functions which are analytic in the closed strip $|\mathrm{Im}(z)| \leq b$.

4. A variational characterization . It was shown by Prager [8] that the sequence $d = (d_k)$ defines a Hilbert space H_d of continuous periodic functions:

$$H_d = \{ f \in L^2_{2\pi} : \sum_{k\neq 0} |(f,e_k)|^2/d_k < \infty \}$$

The inner product of $f,h \in H_d$ is defined by

$$(f,h)_d = \sum_{k\neq 0} (f,e_k)(e_k,h)/d_k + (f,e_0)(e_0,h)$$

H_d is a subspace of the Wiener algebra $A_{2\pi}$ of continuous periodic functions having an absolutely convergent Fourier series. We consider the linear operator U in $L^2_{2\pi}$ which is defined by

$$Uf = \sum_{k\neq 0} \lambda_k (f,e_k)e_k$$

with $|\lambda_k|^2 = 1/d_k \quad (k \neq 0), \quad \lambda_0 = 0$

Obviously, U is a bounded linear operator from H_d to $L^2_{2\pi}$.

Theorem 2 *Assume that* $f \in H_d$. *Then* $Q_n(f)$ *is the unique function in* H_d *which interpolates* f *at the points* t_j , $0 \leq j < n$, *and satisfies*

$$\|UQ_n(f)\| = \min \{ \|Uh\| : h \in H_d,\ h(t_j) = f(t_j)\ (0\leq j<n) \} .$$

Proof. Note first that $g(.-a) \in H_d$, $a \in \mathbb{R}$. This implies

$$V_n^1(g) \subseteq H_d$$

It follows from the basis representation of $V_n^1(g)$ that $Q_n(f)$ has the representation

$$Q_n(f)(t) = c + \sum_{j=0}^{n-1} c_j\, g(t-t_j) , \quad \sum_{j=0}^{n-1} c_j = 0$$

We will show that

(1) $(Uv,UQ_n(f)) = 0 \quad (v \in H_d ,\ v(t_j) = 0 ,\ 0\leq j<n)$.

We have

$$(Uv,UQ_n(f)) = \sum_{j=0}^{n-1} \bar{c}_j\, (Uv,Ug(.-t_j))$$

$$= \sum_{j=0}^{n-1} \bar{c}_j \sum_{k\neq 0} \lambda_k(v,e_k)\bar{\lambda}_k(e_k,g(.-t_j))$$

$$= \sum_{j=0}^{n-1} \bar{c}_j \sum_{k\neq 0} d_k^{-1}(v,e_k)d_k \exp(ikt_j)$$

$$= \sum_{j=0}^{n-1} \bar{c}_j(v(t_j) - (v,e_0)) = 0$$

in view of $v(t_j) = 0$ and $\sum_{j=0}^{n-1} c_j = 0$. Using (1) we get

(2) $\|Uh\|^2 = \|UQ_n(f)\|^2 + \|U(h-Q_n(f))\|^2$

for all $h \in H_d$ with $h(t_j) = f(t_j)$ $(0 \le j < n)$. This implies

$$\|Uh\| \ge \|UQ_n(f)\|$$

On the other hand $\|U(h-Q_n(f))\| = 0$ yields

$$h = c + Q_n(f)$$

with $c \in \mathbb{C}$. Taking into account the interpolation properties of h and $Q_n(f)$ we obtain $c = 0$. This completes the proof of Theorem 2 .

Remark. Note that for the special case $\lambda_k = (ik)^s$, $k \ne 0$, with $s \in \mathbb{N}$ we obtain $Uf = D^s f$ and Theorem 2 yields the variational characterization of the interpolating periodic splines.

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Franz-Jürgen DELVOS,

University of Siegen, Lehrstuhl für Mathematik I

Hölderlinstrasse 3, D-5900 Siegen (West Germany)